Lie Groups, Part 2

Note Title 9/29/2009

Baker-Campbell-Hausdorff philosophy:
recover a "small piece" of G directly
from $\mathfrak{g}$ . .

Algebraic approach: describe algebraically
all distributions on G supported at 1
– ie depending only on the Taylor series
of a function at 1, or equivalently

$$U = \{ \delta : \delta(f) = 0 \text{ if all derivatives of } f \text{ of order} \le n \text{ vanish for } n \text{ sufficiently large} \}$$

– U consists of all differential operators
on G applied to the δ-function
$\delta_1(f) = f(1)$.

Poincaré-Birkhoff-Witt Theorem (distribution version)

As a vector space $U \simeq \operatorname{Sym} \mathfrak{g}$

$$= \bigoplus_{k=0}^{\infty} (\operatorname{Sym}^k \mathfrak{g} = \operatorname{Span}\langle x_1^{i_1} x_2^{i_2} \dots , \textstyle\sum i_j = k\rangle)$$

Proof Pick local coordinates on G near 0
- in particular $\mathfrak{g} \simeq \mathbb{R}^n$.

On $\mathbb{R}^n$ distributions supported at 0
(ie Taylor coefficients) are spanned by

$$\left(\sum \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \dots \right) \cdot \delta_0 . \qquad \blacksquare$$

– note noncanonical!

Extra structure: $\mathcal{U}$ is an associative algebra, via convolution:

Concretely, any function f on G gives a function $\mu^* f$ on $G \times G$ $\quad (\mu : G \times G \to G)$

$\mu^* f(g_1, g_2) = f(g_1 g_2)$.

Given $\delta_1, \delta_2 \in \mathcal{U}$, think of as combinations of Taylor coefficients

$$\delta_1 * \delta_2 (f) = (\text{Taylor coeff } \delta_1 \text{ of } \mu^* f \text{ along } G \times 1)$$
$$\cdot (\text{ " } \quad \text{ " } \quad \delta_2 \text{ " } \quad \text{ " } \quad \text{ " } \quad 1 \times G)$$

— dual to pullback of functions, encodes BCH series implicitly.

Actually $\mathcal{U}$ also has a comultiplication $\mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$: $\mathcal{U}$ is dual vector space to all formal power series (∞ Taylor series, not necessarily convergent) & these have a commutative multiplication (as functions!) $\Rightarrow$ comultiplication on $\mathcal{U}$...

$\rightsquigarrow$ in fact $\mathcal{U}$ is a Hopf algebra

(good topic for projects!)

Comparison: the group algebra of a finite group G. k a ring (e.g. $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$)

kG = k-valued functions on G
= k-valued measures on G
(combos of atomic measures)

- kG has a commutative algebra structure via pointwise multiplication
- kG has an associative algebra structure via convolution:

$$f * h(g) = \sum_{g_1 g_2 = g} f(g_1)\, h(g_2) = \sum_{g'} f(g')\, h(g'^{-1} g)$$

$$= \mu_*(f \boxtimes h): \quad \mu: G \times G \longrightarrow G$$

— sum (pushforward) along fibers of μ: $f \boxtimes h$, $f(g_1)\, h(g_2)$

functions pull back along any map, dually measures push forward.

Concretely: $kG = \langle \delta_g, g \in G \rangle$

$\delta_{g_1} * \delta_{g_2} = \delta_{g_1 \cdot g_2}$, associative but not comm. unless G is ...

$\mathcal{U}$ is a kind of analog of kG for Lie groups — ... captures "G near identity"..

U can be recovered from the Lie algebra $\mathfrak{g}$:

<u>Definition</u> A universal enveloping algebra $U\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$ is a unital associative algebra with a <u>Lie</u> algebra homomorphism $i : \mathfrak{g} \longrightarrow U\mathfrak{g}$ (ie $i(x)i(y) - i(y)i(x) = i[x,y]$) which is universal: given any assoc. A &

$$\begin{tikzcd} \text{Lie alg rep } \mathfrak{g} \arrow[r] \arrow[dr] & A \\ & U\mathfrak{g} \arrow[u, dotted, "\exists!"'] \end{tikzcd}$$

— ie $\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathrm{Forget}(A)) = \mathrm{Hom}_{Ass}(U\mathfrak{g}, A)$

where $\mathrm{Forget}(A)$ means considering A as a Lie algebra only.

— U is the <u>right adjoint</u> to Forget.

$U\mathfrak{g}$, if exists, is <u>unique</u> up to isomorphism: given two such $U\mathfrak{g}$ & $U'\mathfrak{g}$ get maps

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U\mathfrak{g} \arrow[d, "\wr"] \\ & U'\mathfrak{g} \end{tikzcd}$$

which are mutually inverse.

[Ref.: Bump, Lie Groups]

<u>Proposition</u> $U\mathfrak{g}$ exists, & $\mathfrak{g} \to U\mathfrak{g}$ is injective.

<u>Proof / Construction</u> Let $\bigotimes V$ denote the <u>tensor algebra</u> of the vector space V

$\bigotimes V = \bigoplus_{k=0}^{\infty} V^{\otimes k}$. This is an associative algebra with unit $1 \in \mathbb{R} =: V^{\otimes 0}$ & assoc. product $V^{\otimes k} \otimes V^{\otimes j} \xrightarrow{\sim} V^{\otimes k+j}$

In fact it is the <u>free associative algebra</u> on the vector space V:

any vector space map $V \to A$ where A is an associative algebra extends uniquely

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & A \\ \bigotimes V \arrow[ur, dotted] & \end{tikzcd}$$

, ie $\mathrm{Hom}_{Vect}(V, \mathrm{Forget}(A)) = \mathrm{Hom}_{Alg}(\bigotimes V, A)$

via $\varphi(v_1 \otimes v_2 \dots \otimes v_k) = \varphi(v_1) \cdot \varphi(v_2) \cdot \dots \cdot \varphi(v_k)$.

Now let $U\mathfrak{g} = \bigotimes \mathfrak{g} \big/ \langle [x,y] - (x \otimes y - y \otimes x) \rangle$

.... quotient $\bigotimes \mathfrak{g}$ by the ideal generated by all differences of $[x,y] \in \mathfrak{g}$ &

$x\otimes y - y\otimes x \in \mathfrak{g}^{\otimes 2}$ ..... i.e. create the free algebra generated by $\mathfrak{g}$ & subject to the relation that $[x,y]$ is identified with $xy-yx$ ...

Claim: this is a (or "the") universal enveloping algebra: given <u>any</u> map of vector spaces $\mathfrak{g} \xrightarrow{\varphi} A$ (A algebra) we get a homomorphism $\bigotimes \mathfrak{g} \longrightarrow A$.

If φ is a Lie algebra homomorphism then $\varphi\big([x,y] - (x\otimes y - y\otimes x)\big) = 0$

$\Rightarrow$ φ factors through $U\mathfrak{g}$

Proof that $\mathfrak{g} \hookrightarrow U\mathfrak{g}$:

Every $0\neq x \in \mathfrak{g}$ gives a nonzero vector field on G $\Rightarrow$ a nonzero endomorphism of $C^\infty(G)$ — so we have a map

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & C^\infty(G) \\ & U\mathfrak{g} \arrow[u, dotted] \end{tikzcd}$$

which is injective ∎

PBW Theorem Pick an ordered basis $x_1, x_2, \dots x_n$ of $\mathfrak{g}$. Then $U\mathfrak{g}$ has a basis consisting of lexicographically ordered monomials $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$

... ie as a vector space $U\mathfrak{g} \simeq \mathrm{Sym}\,\mathfrak{g}$

... as if free <u>commutative</u> algebra!

idea: given eg $x_2 x_1$, rewrite as $x_1 x_2 + \underbrace{[x_1, x_2]}_{\text{in } \mathfrak{g} = \mathrm{span}\{x_1, \dots x_n\}}$

$$x_2 x_3 x_1 = x_2(x_1 x_3 + [x_1, x_3])$$

$$= x_1 x_2 x_3 + \underbrace{\left([x_1, x_2] x_3 + x_2 [x_1, x_3]\right)}_{\text{quadratic in } \mathfrak{g}}$$

- proceed by induction.

<u>Corollary</u>

$$\begin{tikzcd} U\mathfrak{g} \arrow[rr, "\sim"] & & \mathcal{U} \\ & \mathfrak{g} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

canonical isomorphism extending

.... for this best to identify $\mathcal{U}$ with <u>left invariant</u> differential operators on G via $D \mapsto D \cdot \delta_1 \in \mathcal{U}$ ... can see $\mathfrak{g} \to \mathcal{U}$ is a Lie algebra map.

To see isomorphism: note $U\mathfrak{g}, \mathcal{U}$ have same size! (ie use both PBW theorems....)

$U\mathfrak{g}$ is a big star of representation theory:

given $\mathfrak{g} \longrightarrow \mathfrak{gl}_n\mathbb{R} = M_n\mathbb{R}$, get

$$\mathfrak{g} \rightsquigarrow U\mathfrak{g} \nearrow \text{algebra map}$$

so can identify reps of $\mathfrak{g}$ with reps of the associative algebra $U\mathfrak{g}$

- easier to think about: $U\mathfrak{g}$ is a place where we can write expressions like $x^k y^j + 3x$, $x, y \in \mathfrak{g}$ ....

In any case, this shows we can describe a "very small piece of G" purely algebraically from $\mathfrak{g} \rightsquigarrow U\mathfrak{g} \simeq U$ $\rightsquigarrow U^* =$ formal power series on G.

Real & Complex

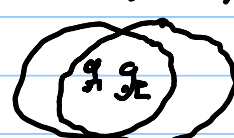
Corollary of BCH Theorem

Every Lie group has a functorial real analytic structure, ie s.t. any homomorphism of Lie groups is real analytic.

Proof (sketch) The exponential map gives a chart on G near 1.

BCH says that the group structure on G is given in this chart by a convergent power series (see Rossmann for linear groups) — ie m is real analytic in this chart.

For any $g \in G$ get chart near g by translating this chart.



transition functions differ by the group structure near 1 $\Rightarrow$ real analytically! ∎

Definition A complex Lie group is a complex manifold with a holomorphic group structure.

Corollary G is a complex Lie group iff $\mathfrak{g}$ is a " " algebra — ie $\mathbb{C}$-vector space with $[,]$ $\mathbb{C}$-linear

Pf G complex $\Rightarrow \mathfrak{g}$ complex: calculus.

Converse follows from BCH: $\mathfrak{g}$ complex $\Rightarrow$ group structure given by convergent power series / $\mathbb{C}$! ∎

<u>Complexification</u> V $\mathbb{R}$-vector space $\Rightarrow V \otimes \mathbb{C}$ $\mathbb{C}$-vector space $\cong V \oplus iV$

$\mathbb{C}$-linear combos of vectors in V.

$\varphi : V \to W$ $\mathbb{R}$-linear $\Rightarrow \varphi_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ $\mathbb{C}$-linear.

$\mathfrak{g}_{\mathbb{R}}$ (real) Lie algebra $\Rightarrow \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ complex Lie algebra.

<u>Def</u> $G_{\mathbb{C}}$ (complex Lie group) is a complexification of G (Lie group) if $\mathrm{Lie}\, G_{\mathbb{C}} = \mathrm{Lie}\, G \otimes \mathbb{C}$. (unique up to isogeny). Conversely we say G is a <u>real form</u> of $G_{\mathbb{C}}$.

U_n <u>NOT</u> complex Lie group!!

Exercise. $GL_n\mathbb{C}$, $(S)O_n\mathbb{C}$, $Sp_n\mathbb{C}$ are. (rarer!)

- $\mathfrak{sl}_n\mathbb{R} \otimes \mathbb{C} = \mathfrak{su}_n \otimes \mathbb{C} = \mathfrak{sl}_n\mathbb{C}$

• $SO(p,q)$ = isometries of $\mathbb{R}^{p+q}$,

$$\|\vec{x}\|^2 = \sum_1^p x_i^2 - \sum_{p+1}^{p+q} x_i^2$$

are all real forms of $SO_n \mathbb{C}$

• $Sp(n)$ (quaternionic orthogonal)

& $Sp(n,\mathbb{R})$ (real symplectic matrices)

both real forms of $Sp(n,\mathbb{C})$.

<u>Proposition</u> G compact & complex

$\Longrightarrow$ G abelian ($\Rightarrow G^0 \simeq \mathbb{T}^{2n}$ some n)

<u>Proof</u> $Ad : G \longrightarrow GL(\mathfrak{g}) \subset \operatorname{End}\mathfrak{g} \simeq \mathbb{C}^{n^2}$

holomorphic $\Longrightarrow$ constant ! ∎

<u>Note</u> G complex $\supset$ H complex subgroup

$\Longrightarrow$ G/H complex manifold (charts given by exp on complement to $\mathfrak{h}$ in $\mathfrak{g}$)

Some homogeneous spaces

- $S^{n-1} \circlearrowleft O(n)$, action is transitive with stabilizer of north pole: $O(n-1)$

$$\Longrightarrow S^{n-1} = O(n)/O(n-1)$$

$$= SO(n)/SO(n-1) : \text{ don't need reflections!}$$

Consequence: $SO(n)$ is connected, since $SO(n-1)$ is by induction & S^{n-1} is.

Note also $O(n), SO(n)$ compact - follows from compactness of S^{n-1} & of $O(n-1), SO(n-1)$.

[Can also use to deduce $\pi_1 SO_n = \mathbb{Z}/2 \quad n \geq 3$
- universal cover is called $Spin_n$]

- $\mathbb{CP}^n \circlearrowleft U(n)$ transitive. Stabilizer is $U(n-1) \times U(1)$ $\quad \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

$$\mathbb{CP}^n = U(n)/U(n-1) \times U(1)$$

$\Longrightarrow$ $U(n)$ connected & compact.
Same for $SU(n)$.

[Can use to deduce $\pi_1 SU(n) = *$]

- $Gr_k(\mathbb{R}^n) = GL_n\mathbb{R}/P_{n,k} = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$

$$= O_n / O_k \times O_{n-k} = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

since every k-plane & its orthogonal complement have orthonormal basis — ie by Gram-Schmidt.

— eg $\mathbb{RP}^n = O_n / O_{n-1} \times O_1$

Similarly $Gr_k \mathbb{C}^n = GL_n\mathbb{C} / P_{n,k}\mathbb{C}$

[$\Rightarrow$ cx. mfld.] $= U_n / U_k \times U_{n-k}$

Consider Gram-Schmidt process:

given $\{v_1, \dots v_n\}$ basis of $\mathbb{C}^n$

$\rightsquigarrow \{u_1, \dots u_n\}$ orthonormal basis:

$$\begin{pmatrix} u_1 = \frac{v_1}{\|u_1\|} & & & 0 \\ u_2 = (\ )\, v_1 + (\ )\, v_2 & & & \\ \dots & & & \\ u_n = (\ )\, v_1 + (\ )\, v_2 + & \dots & + (\ )\, v_n & \end{pmatrix}$$

So any $g = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} \in GL_n\mathbb{C}$

can be written as $g = ub$, $u = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \in U_n$

$b = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ with positive real diagonal entries $v_k/\|v_k\|$, & this factorization is unique:

Theorem $GL_n\mathbb{C} = U_n \cdot B^+$: every g has a unique factorization ub as above.

Let $B = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ all invertible upper triangular matrices .... ie all upper triangular matrices with all diagonal entries nonzero.

$U_n \cap B = T \simeq \mathbb{T}^n$ all diagonal matrices with norm one entries

Corollary $U_n/T \xrightarrow{\simeq} GL_n\mathbb{C}/B$

... this is called the flag manifold, space of flags in $\mathbb{C}^n$ $\quad 0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n$, $\dim V_i = i$. — most important space in representation theory! [note: cx. mfld!]

Partial flags $U_n / U_{i_1} \times U_{i_2} \times \dots \times U_{i_k} = GL_n\mathbb{C}/P_{i_1,\dots,i_k}$

$\qquad\qquad \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \qquad\qquad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$= \{$ flags $V_1 \subset V_2 \subset \dots \subset V_k$, $\dim V_j = i_j \}$.

Corollary $GL_n\mathbb{C}$ is diffeomorphic to $U_n \times \mathbb{R}^k$ ($k = \dim GL_n\mathbb{C} - \dim U_n$)

– in particular homotopic to U_n.

— follows from $GL_n\mathbb{C} = U_n \cdot B^+$

Proposition $GL_n\mathbb{R} = O_n P$, where P = positive definite symmetric matrices

Proof: $g \in GL_n\mathbb{R} \rightsquigarrow$ let $p = (gg^t)^{\frac{1}{2}}$

– unique positive definite $\sqrt{\ }$ of pos. def. symmetric matrix gg^t.

let $o = gp^{-1}$, orthogonal, & $g = op$. ∎

Corollary $GL_n\mathbb{R}$ is diffeomorphic to $O_n \times \mathbb{R}^{\frac{n(n+1)}{2}}$

Pf $P \subset \mathfrak{p}$ symmetric matrices $\simeq \mathbb{R}^{\frac{n(n+1)}{2}}$ is a convex open cone. □

Note: $-(\ )^t$ defines an involution on $\mathfrak{gl}_n\mathbb{R} = \mathfrak{o}_n \oplus \mathfrak{p}$ ± 1 eigenspaces

Same for $-\overline{(\ )}^t$ on $\mathfrak{gl}_n\mathbb{C}$.

Another consequence: $P \simeq GL_n\mathbb{R}/O_n$

homogeneous space: $p \longmapsto$ coset pO_n

Stabilizer of $[p] = pO_n$:

$gpO_n = p(p^{-1}gp)O_n \overset{?}{=} pO_n$

$\iff g \in pO_np^{-1}$: g preserves pos definite inner product $\langle v, w\rangle = v^t P w$.

... P is space of pos def inner products, & any two are $GL_n\mathbb{R}$-related

Similarly $GL_n\mathbb{C}/U_n$ = positive def. hermitian matrices $\mathcal{H}$, ($gl_n\mathbb{C} = u_n \oplus h$) = hermitian inner products on $\mathbb{C}^n$ = subgroups conjugate to U_n.

.... generally G/K parametrizes all subgroups of G conjugate to K.

<u>Theorem</u> G Lie group with $\pi_0 G$ finite

$\Rightarrow \exists$ maximal compact subgroups K.

Any compact subgroup of G is conjugate to a subgroup of K ($\Rightarrow$ any two max cpt are conjugate) & $G \underset{\text{diffeo}}{\cong} K \times \mathbb{R}^k$ some k

Proof for $GL_n\mathbb{R}$ (& $GL_n\mathbb{C}$):

First assume $K \subset GL_n\mathbb{R}$ (compact) preserves some inner product $\langle , \rangle$

By Gram-Schmidt find $g \in GL_n\mathbb{R}$ taking $\langle , \rangle$ to the standard inner product — ie find orthonormal basis for $\langle , \rangle$

$\Rightarrow gKg^{-1} \subset O_n$. (same / $\mathbb{C}$ for U_n).

To find a K-invariant inner product: start with <u>any</u> inner product & average it over K, see below! $\blacksquare$

For any <u>linear</u> group G, we consider $K = G \cap O_n$, compact & any compact subgroup of G is conjugate into K <u>by $g \in GL_n\mathbb{R}$</u>.

<u>In</u> fact (at least for G connected) can force $g \in G$ — requires some more Lie theory (in particular can identify compact subgroups via the Lie algebra...)

Integration on compact Lie groups:

Want to be able to average / integrate over K compact - eg O_n, U_n.

Want $\int_K f(k)\,dk$ s.t

- $\int_K dk = 1$
- $\int_K f(hk)\,dk = \int_K f(kh)\,dk = \int_K f(k)\,dk$ $\forall h \in K$
- positivity

By general calculus on oriented manifolds need to give a volume form

$\omega \in \Gamma(\Lambda^n T^* K)$ ... locally $\omega = f\, dx_1 \wedge \dots \wedge x_n$.

To make integral left-invariant, take ω_L to be a left-invariant differential form — determined by its value $\omega_L|_1 \in \Lambda^n (\mathrm{Lie}\, K)^* \simeq \mathbb{R} \Longrightarrow$ can fix scalar so $\int_K dk = 1$

Why is it right invariant??

Let $\chi : K \longrightarrow \mathbb{R}_+$ ("modular character")

be defined by $\chi(k) =$ ratio of the two left-invariant forms ω_L & $r_k^* \omega_L$ (pullback by right multiplication by k – commutes with left translation)

χ is a character, ie group homomorphism.

In fact $\chi(k) = \det(\mathrm{Ad}\, k^t)$
$= \det \mathrm{Ad}\, k$

– action on k^* is dual to that on k & we're acting on $\Lambda^{top} k^*$.

So $\chi : K \xrightarrow{\mathrm{Ad}} GL(k^*) \xrightarrow{\det} \mathbb{R}_+$

BUT K compact $\Rightarrow \mathrm{Im}\,\chi \subset \mathbb{R}_+$ compact

$\Rightarrow \mathrm{Im}\,\chi = \{1\} \Rightarrow \chi$ trivial

$\Rightarrow \omega_L$ is already bi-invariant!

[Note : G Lie group is called unimodular if it has a bi-invariant volume form if its modular character is trivial. Some [eg classical groups] are, some aren't!]

Exercise. Show the Heisenberg group is not unimodular.

Representation Theory

Def A representation of a Lie group G is a homomorphism $G \xrightarrow[\rho]{} GL(V)$, V a vector space. [we'll only consider V complex]

V, ρ is said to be <u>indecomposable</u> if it can't be decomposed as a direct sum of subrepresentations (G-invariant subspaces)

V, ρ is <u>irreducible</u> if it has no proper invariant subspace.

Note: irred $\Longrightarrow$ indecomposable.
Converse false: $\mathbb{R} \longrightarrow GL_2\mathbb{R}$ $\quad t \longmapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
has $\begin{pmatrix} * \\ 0 \end{pmatrix}$ invariant subspace $\Rightarrow$ not irred, but no invariant complement $\Longrightarrow$ indecomposable. [more generally: <u>Jordan blocks</u> ...]

<u>Schur's Lemma</u>: V_1, V_2 [fin. dim] irreps $\Longrightarrow$ any G-map $f: V_1 \longrightarrow V_2$ is either 0 or an isomorphism.

Any G-map $V_1 \longrightarrow V_1$ is $\lambda \cdot \mathrm{Id}$ $\lambda \in \mathbb{C}$.

<u>Proof</u> $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$ invariant subspaces $\Longrightarrow$ must be 0 or everything!
Any eigenspace $\mathrm{Ker}\,(\lambda I - f)$ invariant $\Rightarrow$ same.
But <u>some</u> eigenspace must be nonzero $\Rightarrow$ $f = \lambda\, \mathrm{Id}$ ∎

Corollary G abelian $\Rightarrow$ any irrep is one dimensional

Proof $\rho: G \to GL(V)$ $\forall g$ $\rho(g)$ commutes with G-action on V
$\Rightarrow \rho(g) = \lambda(g) \cdot Id$ $\forall g$, $\lambda: G \to GL_1\mathbb{R}$
V irred $\Rightarrow$ must be 1-dimensional. $\blacksquare$

Def A unitary rep of G is $\rho: G \to U(V, \langle,\rangle)$ unitary operators on a hermitian vector space

Lemma Any unitary rep is completely decomposable
- any invariant subspace $W \subset V$ has a complementary invariant subspace

Pf Take $W^\perp$. $\blacksquare$

Prop For G compact, any representation is unitarizable ($\exists$ G-invariant hermitian inner product)

Proof Start with any hermitian $\langle,\rangle$ & average over G to make it G-invariant. $\blacksquare$

$\Rightarrow$ Reps of compact Lie groups are completely decomposable. $G \curvearrowright V$ fin dim $\Rightarrow V = \bigoplus V_i$
sum of irreducibles

In particular let $V^G = \{v \in V : g \cdot v = v \ \forall g \in G\}$
<u>invariants</u> (<u>trivial</u> subrepresentation).

Given any $v \in V$ let $v_0 = \int g \cdot v \, dg \in V^G$
For $v \in V^G$, $v = v_0 \Rightarrow$ projection onto invariants.
So $V = V^G \oplus \mathrm{Ker}(\text{projection})$ G-decomposition.

V fin dim $\Rightarrow V = \bigoplus V_i^{\oplus n_i}$ V_i distinct irreps
n_i = <u>multiplicities</u>
= sum of <u>isotypic components</u>.

<u>Prop</u> V_1, V_2 reps of $G \Rightarrow$ so are $V_1^*, V_1 \oplus V_2$, & $V_1 \otimes V_2$

<u>Proof</u> $g \in G$ acts as $\rho_1(g) \otimes \rho_2(g)$ on $V_1 \otimes V_2$.
$\rho_1(g) \oplus \rho_2(g)$ & $\rho_1(g)^t$ on $V_1 \oplus V_2$, V_1^* ∎

Basic questions of rep theory

- classify irreducibles

- decompose natural representations into irreducibles - in particular $L^2(G)$ G compact, $L^2(G/K)$ etc - <u>harmonic analysis</u>

- decompose tensor products of irreducibles into irreducibles: $V_i \otimes V_j = \bigoplus V_k^{\otimes c_{ij}^k} \longrightarrow$ <u>multiplicities</u>

- find concrete descriptions of irreps - e.g. in $L^2(G/H)$ some H

First case $\mathbb{T} = U(1) = SO(2)$.

Irreps: 1-dim & unitary $\Rightarrow \mathbb{T} \xrightarrow{\chi} U(1) \simeq \mathbb{T}$.
differentiate $\Rightarrow \mathbb{R} = \mathrm{Lie}\,\mathbb{T} \longrightarrow \mathbb{R}$ linear map
$\Longleftrightarrow$ multiply by $t \in \mathbb{R}$: $\exp(x) \mapsto \exp(tx)$
$\exp x = e^{2\pi i x} \in U(1)$. Need $1 \mapsto 1$
$\Rightarrow$ need $t = n \in \mathbb{Z} \subset \mathbb{R}$, $\chi_n(x) = e^{2\pi i n x}$
or $z \mapsto z^n$.

<u>Regular representation</u> $\mathbb{T} \circlearrowright L^2(\mathbb{T}) = V$
via left (or right) translation
$\alpha \cdot f(\theta) = f(\theta - \alpha)$

Inside all functions V have characters: functions
f s.t. $f(\theta_1 + \theta_2) = f(\theta_1) f(\theta_2)$: $f \in \mathbb{C}\chi_n$,
$n \in \mathbb{Z}$

$$V \supset \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\chi_n \quad \text{sum of irreducibles:}$$

action $\alpha \in \mathbb{T}$ $\quad \alpha \cdot \chi_n = e^{2\pi i n(x-\alpha)}$
$= e^{-2\pi i n \alpha} \chi_n$

so χ_n lies in χ_{-n} representation!

Fourier series: $\bigoplus \mathbb{C}\chi_n$ is <u>dense</u> in $L^2(\mathbb{T})$

Given $f \in L^2(\mathbb{T})$ let $\hat{f}(n) = \langle f, \chi_n \rangle$
$= \int f(\theta) e^{2\pi i n \theta} d\theta$: component of f
in χ_n direction = projection to χ_{-n}
subrepresentation.

$f(\theta) = \sum \hat{f}(m) \chi_m$ convergent series

More generally let V be any fin dim vector space (in fact same will hold for any locally convex topological vector space, eg Hilbert space....)

Let $V_n = \{ v \in V : g \cdot v = \chi_n(g) \cdot v \}$

χ_n - isotypic component

$v \in V \longmapsto v_n = \int_{\mathbb{R}} \left(e^{-2\pi i n x} \right) v \, dx$

Exercise Use G-invariance of integral to show $v_n \in V_n$.

$\Rightarrow v = \sum v_n$, v_n are orthogonal projections to V_n. & $V = \oplus V_n \simeq \oplus \mathbb{C}\chi_{-n}^{\oplus \dim V_n}$

(∞ dim case: $\sum v_n \longrightarrow v$ converges & $V = \widehat{\oplus V_n}$ closure)

Another way to see V_n: any vector $v \in V_n$ lives in a 1-d subrep of V isom. to $\mathbb{C}\chi_{-n}$.

$\Rightarrow$ a homomorphism of representations

$$V_n \simeq \mathrm{Hom}_G(\mathbb{C}\chi_{-n}, V)$$

$$\varphi(\chi_{-n}) \longleftarrow \varphi$$

Also $V_n \simeq (V \otimes \mathbb{C}\chi_n)^G$ shifted invariants

- change action by factor χ_{-n} & take invariants....

From characters to matrix elements:

G abelian, V irrep $\Rightarrow$ V 1-dim, unitary

$\chi : G \to U(V) \simeq U(1)$ is a function on G

which is a character, $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$
& norm one.

$\chi(g)$ = ratio of any $0 \neq v \in V$ to $g \cdot v$.

- continuous function on G,
will be L^2 for G compact.

$\mathbb{C}\chi \subset C(G)$ or $L^2(G)$
will be invariant under translation &
G acts on it by dual character.

Fourier theory Any function on G abelian
can be decomposed into characters.

$G = \mathbb{R}$: characters are $e^{2\pi i t x}$ $t \in \mathbb{R}$
& this gives Fourier transform...

For G compact abelian $\Rightarrow$ $G \simeq \mathbb{T}^n$
get characters $\leftrightarrow$ $\vec{i} \in \mathbb{Z}^n$: prescribe
on each factor ... $L^2(\mathbb{T}^n) \simeq \bigoplus \mathbb{C}\chi_{\vec{i}}$
Hilbert space direct sum.

G any abelian Lie group (or locally compact
topological group) $\Rightarrow$ Pontrjagin duality
... general version of Fourier transform...

We'll go in a different direction: <u>compact nonabelian groups</u> $\rightsquigarrow$ nonabelian harmonic analysis.

<u>Matrix elements</u> V any G-rep (fin dim)

$\Rightarrow$ realize inside $L^2(G)$:

given $v \in V$, $v^* \in V^*$ let $f_{v,v^*} \in C(G) \subset L^2(G)$

be $f_{v,v^*}(g) = \langle v^*, g \cdot v \rangle = \langle g^{-1^t} v^*, v \rangle$

(If V is unitary $\Rightarrow$ identify $V^* \simeq \overline{V}$:

Same vector space as V but G action

is complex conjugated: $\langle v_1, g v_2 \rangle = \langle \bar{g} v_1, v_2 \rangle$)

So we get a map

$$V \otimes V^* \simeq V \otimes \overline{V} \longrightarrow C(G)$$
$$v, v^* \longmapsto f_{v,v^*}.$$

... for any fixed v^* get $V \longrightarrow C(G)$.

Any vector space $\left\{ \begin{array}{l} V \otimes V^* = \mathrm{End}(V) \quad \text{matrices on } V \\ v \otimes v^* \longleftrightarrow \text{elementary matrix} \\ \qquad A \cdot w = v^*(w) \cdot v \end{array} \right.$

In particular have a distinguished elt of $V \otimes V^*$,

$\mathrm{Id} \in \mathrm{End}\, V$! get a distinguished

function on G denoted χ_V – what is it?

In basis e_i of V, $Id = \sum e_i \otimes e_i^*$

$\Rightarrow \chi_V = \sum f_{e_i, e_i^*}$, $f(g) = \sum \langle e_i^*, g e_i \rangle$

ie $\chi_V(g) = tr\, \rho(g)$: the <u>character</u> of the representation (V, ρ)

(note: not character in sense of hom. $G \to GL_1 \ldots$)

e.g $\chi_V(1) = tr\, Id_V = \dim V$.

$\chi_V(h^{-1} g h) = \chi_V(g)$: <u>class function</u>

Observe: $L^2(G)$ is actually a unitary rep of $G \times G$, acting by left & right translation!

$h_1 \times h_2 \cdot f(g) = f(h_1^{-1} g h_2)$

So is $V \otimes V^*$: $\rho(h_1, h_2)(v \otimes v^*)$
$= \rho(h_2) v \otimes \rho(h_1) v^*$.

Matrix elements $V \otimes V^* \longrightarrow L^2(G)$ is a $G \times G$-map:

$\langle v^*, h_1^{-1} g h_2 v \rangle$

$= \langle h_1 v^*, g \cdot h_2 v \rangle$

<u>V irreducible</u> : <u>this map is injective</u>:

Calculate $\mathrm{End}(V \otimes V^*)^{G\times G}$

$$= \mathrm{End}\, V^G \otimes \mathrm{End}\, V^{*\,G} \simeq \mathbb{C}$$

$\Rightarrow$ $V \otimes V^*$ is an irreducible $G\times G$-rep

& map $V \otimes V^* \longrightarrow L^2(G)$ nonzero

$$\mathrm{Id}_V \longmapsto \chi_V \neq 0$$

If we consider $\mathrm{End}\, V \simeq V \otimes V^*$ as rep of <u>diagonal</u> $G \subset G\times G$, then it's not irreducible but has 1-dim invariant subspace $\mathbb{C}\cdot \mathrm{Id}_V = (\mathrm{End}\, V)^G$: this is just statement that χ_V is a class function!

Also for any fixed $v^* \in V^*$ get an embedding $V \hookrightarrow L^2(G)$ as (left) G-rep.

Conversely given <u>any</u> $V \xrightarrow{\varphi} L^2(G)$ G-map get $v^* \in V^*$ by $v^*(v) = \varphi(v)|_1$.

So these are <u>all</u> the appearances of V in $L^2(G)$: appears with multiplicity $\dim V$.

Def For V irrep, W rep' of G, the V-<u>isotypic component</u> W_V is the sum of all copies of V in W, ie the image of

$$V \otimes \mathrm{Hom}_G(V, W) \longrightarrow W$$

Thus $C(G)_V \simeq L^2(G)_V = V \otimes V^*$

<u>Orthogonality relations</u> f_{W,w,w^*} & f_{V,v,v^*} two matrix elements are <u>orthogonal</u> unless $V \simeq W$

<u>Proof</u>: Suppose nonzero $\Rightarrow$ define map $V \longrightarrow W$, project $V \xhookrightarrow[v^*]{} L^2(G)$ onto $W \xhookrightarrow[w^*]{} L^2(G)$, get nonzero map - contradiction! $\blacksquare$

<u>Lemma</u> V irrep $\Rightarrow \exists!$ invariant inner product up to scalar

<u>Pf</u> Compact $G \Rightarrow$ existence.
Uniqueness: any $\langle , \rangle$ induces G-map $V \longrightarrow \bar{V}^*$, must be unique / scalar. $\blacksquare$

$\Rightarrow V \otimes V^*$ has unique inner product - but it obtains one from inclusion in $L^2(G)$

$$\Rightarrow \langle f_{V,v_1,v_1^*}, f_{V,v_2,v_2^*} \rangle = c_V \langle v_1, v_2 \rangle \overline{\langle v_1^*, v_2^* \rangle}$$

for some c_V.

Evaluate c_V: take orthonormal basis e_i

$f_{V,e_i,\bar{e}_j}(g) = M_{ij}(g)$ $\quad$ ij entry of matrix of g

$$\Rightarrow c_V \delta_{ik}\delta_{jl} = \int_G \overline{M_{ij}(g)} M_{kl}(g)\, dg$$
$$= \int_G M_{ji}(g^{-1}) M_{kl}(g)\, dg$$

Put $i=k$ & sum over i:

$$\int M_{jl}(1)\, dg = c_V \dim V \delta_{jl}$$
$$\Rightarrow \boxed{c_V = \frac{1}{\dim V}}$$

Corollary V, W irreps, $\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V = W \\ 0 & \text{otherwise} \end{cases}$

$\cdots\cdots\!>$ V determined by its character...

Corollary $\underbrace{\left(\bigoplus_{V \text{ irrep}} V \otimes V^*, \frac{1}{\dim V} \langle\rangle\langle\rangle \right)}_{\circ} \subset \left(L^2(G), \int_G \overline{(\ )}(\ ) \right)$

def: $\mathbb{C}[G]$ algebraic fns on G

Prop $\mathbb{C}[G] \subset C(G)$ subalgebra of continuous functions on G

Proof $f_{W,w,w^*} + f_{V,v,v^*} = f_{W\oplus V, w\oplus v, w^*\oplus v^*}$

& $f_{W,w,w^*} \cdot f_{V,v,v^*} = f_{W\otimes V, w\otimes v, w^*\otimes v^*}$ ∎

Example $\mathbb{T}$: $\mathbb{C}[\mathbb{T}] = \bigoplus z^n \quad n \in \mathbb{Z}$

$\simeq \mathbb{C}[z, z^{-1}]$ polynomial functions on complexification $\mathbb{C}^*$ .....

Can characterize $\mathbb{C}[G]$ intrinsically :

Def For V any rep of G, V^{fin} = all vectors contained in some finite-dim subrepresentation

Lemma $\mathbb{C}[G] = C(G)^{fin}$ under left translation action

Pf $\subset$ clear. $\supset$: $f \in W \subset C(G)^{fin}$ $\Rightarrow$ decompose into irreps & we know all appearances of those are matrix elts. ∎

Nonabelian Fourier Theory: Peter-Weyl Theorem

Theorem $\mathbb{C}[G] \subset C(G) \subset L^2(G)$ dense for topology of uniform convergence / L^2 topology.

Corollary 1 Any compact Lie group G is a subgroup of a unitary group U_n ($\Rightarrow$ matrix gp)

Pf $\mathbb{C}[G] \subset C(G)$ dense $\Rightarrow$ algebraic fns separate points in G $\Rightarrow$ any $g \in G$ acts faithfully in some f.d. rep V_g.

Consider $\operatorname{Ker}(G \to \operatorname{Aut}(V_{g_1} \oplus \dots \oplus V_{g_k}))$
for larger & larger finite subsets of G
- descending chain of Lie subgroups.

$\Rightarrow$ must eventually be trivial (check first on Lie algebra then on fin many components!) $\blacksquare$

Note: <u>Corollary 1 $\Rightarrow$ Theorem</u>: $G \subset M_n\mathbb{C}$
$\Rightarrow$ polynomials in coord fns dense in continuous fns on compact subset G
(Stone–Weierstrass Theorem)

<u>Plancherel Theorem</u> $\|f\|^2 = \sum_{V \text{ irrep}} \frac{1}{\dim V} \|f_V\|^2$

f is L^2 $\longleftrightarrow$ $\{\|f_V\|\}$ square summable
f is C^∞ $\longleftrightarrow$ $\|f_V\|$ rapidly decreasing
f is C^ω (real analytic) $\longleftrightarrow$ $\|f_V\|$ exponentially decreasing

f in algebraic $\longleftrightarrow$ f_V <u>finite</u>

<u>Corollary</u> $C(G/H) \simeq \bigoplus_{V \text{ irrep}} V \otimes \bar{V}^H$

- Fourier theory for homogeneous spaces

Corollary Characters form an orthonormal basis for the Hilbert space of L^2 class functions.

Proof of Peter-Weyl (sketch)

First note that $V \otimes V^* \cong \operatorname{End} V$ has a nondegen. inner product, G-invariant

$\langle A, B \rangle = \operatorname{Tr}(\bar{A} B)$ — for elementary matrices $v \otimes v^*$ & $w \otimes w^*$ this is what we wrote as $\langle \overline{v^*, w^*} \rangle \langle v, w \rangle$.

We need to show functions of form

$g \mapsto \operatorname{Tr}(A \rho(g))$ $\qquad \rho : g \to \operatorname{Aut} V$, $A \in \operatorname{End} V$

uniformly approximate all functions.

Idea: want to find fin dim G-invariant subspaces of $C(G)$

— look at eigenspaces of operators $C(G) \circlearrowleft$ commuting with G-action ($\Rightarrow$ act by scalar on each irreducible!)

Represent operators by <u>integral transforms</u> — continuous analog of matrices:

X, Y finite sets, $A \in \operatorname{Fun}(X \times Y)$ gives operator $\operatorname{Fun}(X) \longrightarrow \operatorname{Fun}(Y)$ via

$$f \longmapsto \pi_{2*}(A \cdot \pi_1^* f)$$

$$\text{"} = \int_{\pi_2} A\, \pi_1^* f \text{"}$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

... $\mathrm{Fun}(X) \simeq \mathbb{C}^n$, $\mathrm{Fun}\, Y \simeq \mathbb{C}^m$

$\mathrm{Fun}\, X \times Y \simeq \mathrm{Mat}_{n \times m}$

& this is $\simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

X locally compact Hausdorff spaces.

Riesz representation: any linear functional $C_c(X) \to \mathbb{C}$ given by a measure on X

$$f \longmapsto \int f \, d\mu.$$

Integral transforms:

$$\begin{tikzcd} & \overset{k}{X \times X} \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

$$k * f(y) = \int k(x,y) f(x)\, dx$$

[Schwartz kernel theorem: if we allow k to be a distribution realize any continuous operators on C_c^∞ eg this way...]

So we'll think of $C(G \times G)$ as operators $C(G) \to C(G)$.

G acts on $G \times G$ diagonally $\rightsquigarrow$
$(g \cdot k)(x,y) = k(g^{-1}x, g^{-1}y)$

k is G-invariant $\iff$ $k * : C(G) \to C(G)$
is a G-map.

Now apply spectral theory of self-adjoint compact operators . . . .

<u>Theorem</u> k hermitian & $u \in C(G)$
$\Rightarrow$ 1. $\int k(x,y)\, u(y)$ can be uniformly approximated by a <u>finite</u> linear combo of eigenfunctions of k with $\neq 0$ eigenvalues.

2. $\lambda \neq 0 \rightsquigarrow$ eigenspaces V_λ are finite dimensional - in fact $\sum |\lambda_i|^2$ converges!

Group algebra revisited

$C(G)$ is an associative algebra under convolution:

$$f_1 * f_2\,(h) = \int_G f_1(g)\, f_2(g^{-1}h)\, dg.$$

$f \in C(G) \rightsquigarrow k_f \in C(G \times G)$ kernel
$k_f(x,y) = f(x^{-1}y)$.

$$\Rightarrow f_1 * f_2 = k_{f_1} * f_2 \quad : C(G) \to C(G\times G) \to \mathrm{End}\, C(G).$$

Role of group algebra: functions on G are continuous / smeared linear combos of elements of G $\Rightarrow$ give linear operators in any rep V of G:

$$f \in C(G) \longmapsto T_f : V \to V$$

$$T_f v = \int_G f(g)\, g\cdot v \, dg$$

This makes V into a $C(G)$-module:

Exercise: $T_{f_1 * f_2} = T_{f_1} \circ T_{f_2}$.

- Note: T_f self-adjoint $\Longleftrightarrow$ f hermitian: $f(g^{-1}) = \overline{f(g)}$

Note: for $f \in \mathbb{C}[G] = C(G)^{fin}$,

$T_f v \in V^{fin} \quad \forall v \in V$, since

$g T_f v = T_{gf} v$ & $\{g\cdot f : g \in G\}$

span a fin-dim representation (mapping G-invariantly into V) ...

Would like to say $g = T_{\delta_g}$ δ-function of g

– not in $C(G)$ though. In particular no identity operator T_{δ_1}

<u>But</u> can approximate Id operator by approximating δ_1: take bump function $b \geq 0$, $\int b = 1$, Supp $b = \delta$-ball around 1:

<u>Lemma</u> Any $f \in C(G)$ uniformly approximated by fns of form $v(x) = \int_{y \in G} k(x,y) f(y)$ with k real symmetric & G-invariant.

<u>Pf</u>: find δ s.t. $|f(x) - f(y)| < \varepsilon$ for $x^{-1}y$ in δ-ball near 1.

Take $k(x,y) = \mu(x^{-1}y)$, μ a bump function supported on δ-ball around 1 ($\mu \geq 0$, $\int \mu = 1$), $\mu(x^{-1}) = \mu(x)$

– ie average f over δ ball around each point, doesn't do too much:

$$|\mu(x^{-1}y) f(x) - \mu(x^{-1}y) f(y)| \leq \varepsilon \mu(x^{-1}y)$$

everywhere ... integrate over y:

$$\Big|f(x) - \underbrace{\int \mu(x^{-1}y) f(y)\, dy}_{k * f(x)}\Big| \leq \varepsilon$$

$\blacksquare$

Now: k_f G-invariant ($\Longleftrightarrow$ f class function)

$\Rightarrow$ T_f on $C(G)$ commutes with left G-action on $C(G)$ $\Rightarrow$ eigenfunctions are in $C(G)^{fin}$

So any f is approximated by a finite sum eigenfunctions hence by algebraic functions! $\blacksquare$

In fact this proves more:

Theorem V any representation (loc. convex topological vector space) $\Rightarrow$

$$V^{fin} = \bigoplus_{\substack{W \text{ f.d.im} \\ \text{irrep}}} V_W \text{ is dense in } V.$$

Pf Can approximate δ_1 by $f \in C(G)$

$\Rightarrow$ by Peter-Weyl by $f \in C(G)^{fin}$

So any $v \in V$ approximated by $T_f \cdot v \in V^{fin}$. $\blacksquare$

Corollary All irreps of G compact are finite dimensional!

eg $G = \mathbb{T}$ approximate $\delta_1 \text{ "="} \sum_{n \in \mathbb{Z}} e^{2\pi i n x}$

by finite sums – this says any function is approximated by Fourier series, in <u>any</u> function space, or <u>any</u> rep of $\mathbb{T}$.

Algebra structure on $C(G)^{fin}$:

<u>Prop</u> $(\mathbb{C}[G], *) \xrightarrow{\sim} \left(\bigoplus_{W \text{ irrep}} \operatorname{End} W, \text{composition}\right)$

$f \longmapsto T_f$

algebra isomorphism. – full picture of $\mathbb{C}[G]$ as Hopf algebra!