Lie Groups, Part III

Note Title 10/21/2009

Representations of SU_2

Let's describe all irreps of SU_2.
We'll do this twice: group theoretically & Lie algebra theoretically.

V f.d. rep of SU_2: study first as rep of $T \subset SU_2$ $\quad T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cong \mathbb{T}$

maximal torus of SU_2: all matrices preserving fixed axis in $\mathbb{R}^3$.

$$\Rightarrow \quad V \cong \bigoplus_{n \in \mathbb{Z}} V_n$$

(finite sum).

isotypic components for $\mathbb{T}$ action } weight spaces - w/ n.

What can we say about the V_n?
- use rest of group!

$H \subset G$ a subgroup

$N(H)$ its normalizer in G acts on H

$n : H \to H \qquad h \mapsto n^{-1} h n.$

... in fact $N(H)/Z(H)$ acts, centralizer

$\Rightarrow N(H)/Z(H)$ acts on the collection of all H-reps: W H-rep $\longrightarrow W^{(n)}$ new

H-rep: same vector space but
$\rho^{(n)}(h) = \rho(n^{-1} h n)$: twist action by symmetries of H.

V G-rep $= \bigoplus V_w$ H decomposition

$\rightsquigarrow$ $n \in N(H)$ takes $V_w \subset V$ to

$V_{w^{(n)}}$: $\rho(h)\rho(n)v = \rho(n)\rho(n^{-1}hn)v$

$= \rho(n)\rho^{(n)}(h)v$

$\Rightarrow \rho(n) : V_w \xrightarrow{\sim} V_{w^{(n)}}$

Our case : $N(H)/H = Z(H) \cong \mathbb{Z}/2$

<u>Weyl group</u> of SU_2

$\rightsquigarrow$ can see in SO_3: $180°$ rotations in perpendicular axis normalize rotations around a fixed axis, conjugation reverses the direction of rotation

Or in $SL_2\mathbb{C}$: T conjugate to matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in U(1) \subset \mathbb{C}^*$

Normalizer given by: $\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$, up to T

just need $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, acts as $a \mapsto a^{-1} = \bar{a}$.

<u>Corollary</u> $V_n \cong V_{-n}$!

acting by $-1 \in \mathbb{Z}/2$ takes $e^{2\pi i n z}$ eigenspace to $e^{-2\pi i n z}$ eigenspace.

What patterns are allowable?
study character $\chi_V|_T$ as fn on T.

$\chi_V(t) = \mathrm{tr}\,\rho(t)$

but $\rho = \mathrm{diag}(q^{n_1} q^{n_1} \quad q^{n_1} \quad q^{n_2} q^{n_2} \dots)$
on V_{n_1} $\quad V_{n_2}$

etc so $\qquad q = e^{2\pi i n t}$

$\chi_V = \sum (\dim V_n) q^n$ integer finite

Laurent polynomial in q
& symmetric in $q \longleftrightarrow q^{-1}$.

Moreover any $g \in SU_2$ is conjugate
into T: any rotation has an axis!

SO since χ is a class function,
χ is determined by $\chi|_T$!

In fact we find $\underset{\text{conjugation}}{SU_2/SU_2} \cong \underset{\substack{\text{circle mod} \\ q \longleftrightarrow q^{-1}}}{T/W}$

so irreducible characters form a basis for
functions on $\mathbb{T}$ symmetric wrt $q \longleftrightarrow q^{-1}$.

So all a character is is the list of dimensions
$\dim V_n$, encoded as function $\sum \dim V_n \cdot q^n$.
("Formal character")

How do we determine the basis of irreducible characters? $\rightsquigarrow$ they're orthonormal for L^2 inner product on G .....

need to figure out what that gives on class functions, i.e in variable q .....

$\rightsquigarrow$ Weyl integration formula

First we'll prove a generalization of the fact that every $g \in SU_2$ has an axis $\longleftrightarrow$ has a fixed point in $S^2 = SU_2/U(1)$:

Prop Any $g \in GL_n\mathbb{C}$ fixes some flag in $\mathbb{C}^n$, ie has a fixed point in $\mathcal{B} = GL_n\mathbb{C}/B$. (flag manifold)

Proof: Find eigenvector v_1, then eigenvector in $\mathbb{C}^n/\mathbb{C}v_1$, etc. $\blacksquare$

Corollary Any unitary matrix is diagonalizable

Pf $\mathcal{B} \simeq U(n)/T$. = subgroups conjugate to T = diagonal matrices.

If g fixes $[T'] \in \mathcal{B} \Rightarrow g \in T'$

$\Rightarrow$ conjugate into T. $\blacksquare$

[Theorem G compact Lie group $\Rightarrow$
every $g \in G$ contained in a maximal torus $\mathbb{T}^n \simeq T \subset G$ & any two max. tori are conjugate. $n =: \text{rank}(G)$.]

So any class function on $U(n)$ is determined by its restriction to T.

In fact $N(T)/T =: W$ Weyl group
$\simeq S_n$ permutation matrices.

So class functions on $U(n) \longleftrightarrow$ symmetric functions on T.

Write as function of $q_1, \ldots, q_n$ coords on $\mathbb{T}^n$.

Weyl integration formula

Let $\Delta = \prod_{i<j} (q_i - q_j) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & \cdots & q_n \\ q_1^2 & & \cdots & & q_n^2 \\ q_1^{n-1} & & \cdots & & q_n^{n-1} \end{vmatrix}$

Vandermonde determinant

$$\Delta \bar{\Delta} = \prod_{i<j} |q_i - q_j|^2$$

f, g L^2 class functions on $U(n)$

$$\langle f, g \rangle_G = \int_G \bar{f} g \, dh = \frac{1}{n!} \int_T \bar{f} g \bar{\Delta} \Delta \, dt$$

$$= \frac{1}{n!}\langle f\Delta, g\Delta\rangle_T = \langle f\Delta, g\Delta\rangle_{\{q_1 \le q_2 \le \dots q_n\} \subset T}$$

remove $n!$ overcounting by picking

Weyl alcove : fundamental domain for $S_n \curvearrowright T$.

$$\Longleftrightarrow \quad \int_G f\, dg = \frac{1}{n!}\int_T f \prod |q_i - q_j|^2\, dq_1 \dots dq_n$$

$\Longleftrightarrow$ volume of conjugacy class through $q_1, \dots, q_n$ is $\prod |q_i - q_j|^2$

Proof: Consider

$$\begin{aligned} T \times G/T &\longrightarrow G \\ t \quad gT &\longmapsto gtg^{-1} \end{aligned}$$

sweep out all tori in G (surjective map).

Let $J(t)$ = Jacobian of this map (indep of gT by invariance).

Map generically $n!$ to 1 (permutation of eigenvalues)

$$\Rightarrow \int_G f = \frac{1}{n!}\int_{T\times G/T} f(gtg^{-1}) J(t)\, dt\, d(gT)$$

$$= \frac{\mathrm{vol}(G/T)}{n!}\int_T f(t) J(t)\, dt$$

Calculate J infinitesimally:

write tangent to G/T at gT as $\mathfrak{g}/\mathfrak{t} \simeq \mathfrak{t}^\perp$.

$g=1$

Now we vary t, gT infinitesimally to

$$t(1+\xi) \qquad 1+\eta$$
$$(\xi \in \mathfrak{t}) \qquad (\eta \in \mathfrak{t}^\perp)$$

& calculate change to gtg^{-1}:

$$t^{-1}\big((1+\eta)t(1+\xi)(1-\eta) - t\big)$$
$$= t^{-1}(t\xi + \eta t - t\eta) = \xi + (t^{-1}\eta t - \eta)$$
$$\in \mathfrak{t} \oplus \mathfrak{t}^\perp = \mathfrak{g}$$

So $J(t) = \det(\eta \mapsto t^{-1}\eta t - \eta)$

$= \det(\text{Ad action of } t^{-1} \text{ on } \mathfrak{t}^\perp - \text{Id})$

Complexify: $\mathfrak{t}^\perp \otimes \mathbb{C} = \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix}$

basis E_{jk} $j \neq k$
elementary matrices

$t = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}$ $\Rightarrow$ E_{jk} eigenvector with eigenvalue $\frac{q_k}{q_j}$ of $\mathrm{Ad}\, t^{-1}$

$$\Rightarrow J(t) = \prod_{j \neq k} \left(\frac{q_k}{q_j} - 1\right) = \prod_{j<k} |q_j - q_k|^2$$

So we're done mod $\mathrm{Vol}(G/T)$ .. but check $1 = \int_G dg = \frac{1}{n!}\int_T \Delta\bar{\Delta}\, dq_1 \dots dq_n = \frac{1}{n!}\int \prod |q_i - q_j|^2\, dq$ $\blacksquare$

$\Rightarrow$ we get all irreducible characters of $U(n)$ by finding orthonormal basis of skew-symmetric functions $\chi\Delta$ ...

take $\chi\Delta = \sum_{w\in W} (-1)^{\ell(w)} q_1^{w(i_1)} \cdots q_n^{w(i_n)}$

for some $i_1, \dots i_n$

$$\Rightarrow \chi = \frac{\sum_{w\in W} (-1)^{\ell(w)} q_1^{w(i_1)} \cdots q_n^{w(i_n)}}{\prod_{i<j} q_i - q_j}$$

$U(2)$: $\dfrac{q_1^{n_1} q_2^{n_2} - q_1^{n_2} q_2^{n_1}}{q_1 - q_2}$ $\begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix}$

Restrict to SU_2 : $q_1 = q_2^{-1} = q$

$$\boxed{\chi_n = \frac{q^n - q^{-n}}{q - q^{-1}}}$$

$$= q^n (1 + q^{-2} + q^{-4} + \dots)$$
$$- q^{-n} (1 + q^{-2} + q^{-4} + \dots)$$

$$= q^n + q^{n-2} + \dots + q^{2-n} + q^{-n}$$

weight spaces go in jumps of 2 from a highest weight n to lowest weight $-n$.

.... irrep of SU_2 with highest weight n.

Lie algebraic approach

V irrep of $SU_2 \Rightarrow$ of $\mathfrak{su}_2$.

But V is a complex vector space

$\Rightarrow$ has an action of $\mathfrak{su}_2 \otimes \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$.

$\Rightarrow$ eg has an action of $\mathfrak{sl}_2\mathbb{R}$

$\Rightarrow$ of group $SL_2\mathbb{R}$ (or some cover.... in fact do get $SL_2\mathbb{R}$ ....)

[... Weyl unitary trick: V any f.dim rep of $SL_2\mathbb{R} \Rightarrow$ completely decomposable!

.... get action of $\mathfrak{sl}_2\mathbb{R} \rightsquigarrow \mathfrak{sl}_2\mathbb{C} \rightsquigarrow \mathfrak{su}_2 \rightsquigarrow SU_2 \Rightarrow$ can find invariant complements!

... applies to any of the groups $SL_n\mathbb{C}, GL_n\mathbb{C}, (S)O_n\mathbb{C}, Sp_n\mathbb{C}$, .. & their real forms.]

Anyway... write $\mathfrak{sl}_2\mathbb{C} = \mathrm{Span}(e, h, f)$

$[h,e] = 2e$ $\quad [e,f] = h$ $\quad \begin{pmatrix} & 1 \\ & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & \\ 1 & \end{pmatrix}$

$[h,f] = -2f$

$\mathbb{C}h = \mathrm{Lie}(\text{diagonal matrices})$ – makes sense to try to diagonalize h on a rep V of SU_2 (note: h diagonalized on $\mathfrak{sl}_2\mathbb{C}$!)

Suppose $h \cdot v = \lambda v$ eigenvector $\Rightarrow$

$$h \cdot (ev) = e(hv) + 2ev$$
$$= (\lambda+2)\, ev \quad : \text{ eigenvector}$$

with eigenvalue $\lambda+2$.

$$h \cdot (fv) = f(hv) - 2fv = (\lambda - 2) v.$$

So get tower of eigenvectors

$$\begin{array}{ccccccccccccc} \cdots & \leftarrow & f^2v & \leftarrow & fv & \leftarrow & v & \rightarrow & ev & \rightarrow & e^2v & \rightarrow & e^3v \; \cdots \\ & & \lambda-4 & & \lambda-2 & & \lambda & & \lambda+2 & & \lambda+4 & & \end{array}$$

V finite dimensional $\Rightarrow e^{k+1} v = 0 \quad k \gg 0$

... ie eventually find <u>highest weight vector</u>

$0 \neq w$: $\quad hw = \nu w \quad (\nu = \lambda + 2k \ldots)$

& $ew = 0$. top of ladder!

Now apply ladder down:

$$\cdots\cdots\cdots \; f^2w \quad fw \quad w$$

after some number of steps get 0:

$$\begin{array}{c} \left[\begin{array}{ccccc} f^k w & f^{k-1} w & \cdots & \cdots & w \end{array} \right] \\ \begin{array}{ccccc} \nu - 2k & \qquad & & \qquad\qquad\qquad & \nu \end{array} \end{array}$$

<u>Claim</u> $ef^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$
(when applied to any vector, or in universal enveloping algebra).

<u>Pf</u> Induction on n.
$n=0$ $ef = fe + h$
Assume for n. $ef^{n+2} = \left(f^{n+1}e + (n+1)f^n(h-n)\right)f$
$= f^{n+2}e + f^{n+1}h + (n+1)f^{n+1}(h-n-2)$
$= f^{n+2}e + (n+2)f^{n+1}(h-n-1)$ $\blacksquare$

<u>Corollary</u> Our chain is a <u>subrepresentation</u>:
$$ef^{n+1}w = f^{n+1}\underbrace{ew}_{0} + (n+1)f^n(h-n)w$$
$$= (\nu-n)(n+1)f^n w:$$
up to scalar e takes our basis vectors to other basis vectors!

Also get constraint on ν:
$$0 = ef^{k+1}w = (k+1)(\nu-k)\underbrace{f^k w}_{\neq 0}$$
$\Rightarrow$ <u>$\nu = k$</u>. So our rep is
$$[-k \quad 2-k \quad \ldots\ldots \quad k-2 \quad k]$$
integer & symmetric around 0!

- recover character $\chi_k = \dfrac{q^k - q^{-k}}{q - q^{-1}}$.

Concrete realization:

$G \curvearrowright V \Rightarrow G \curvearrowright \mathrm{Sym}^n V$ via $g \cdot (v_1 v_2 \dots v_n)$
$= (gv_1)(gv_2)\dots(gv_n)$

$\Rightarrow \mathfrak{g}$ acts via $x \cdot (v_1, \dots v_n) =$
$\frac{d}{dt}\Big|_{t=0} e^{tx} \cdot (v_1 \dots v_n) =$

$(xv_1) v_2 \dots v_n + v_1 (xv_2) \dots v_n + \dots + v_1 \dots (xv_n)$
derivation.

$\mathfrak{sl}_2\mathbb{C} \curvearrowright \mathbb{C}^2 = \mathrm{Span}\{x, y\}$

$\Rightarrow$ acts on $\mathrm{Sym}^n \mathbb{C}^2 = \mathrm{Span}\{x^n, x^{n-1}y, \dots y^n\}$

$hx = -x \quad hy = +y \qquad ey = 0 \quad ex = y$
$fx = y \quad fy = 0$

- can write as $e = y\frac{d}{dx} \qquad f = x\frac{d}{dy}$
$h = y\frac{d}{dy} - x\frac{d}{dx}$

$$x^n \underset{e}{\overset{f}{\rightleftarrows}} x^{n-1}y \underset{e}{\overset{f}{\rightleftarrows}} x^{n-2}y^2 \quad \dots \quad xy^{n-1} \underset{e}{\overset{f}{\rightleftarrows}} y^n$$

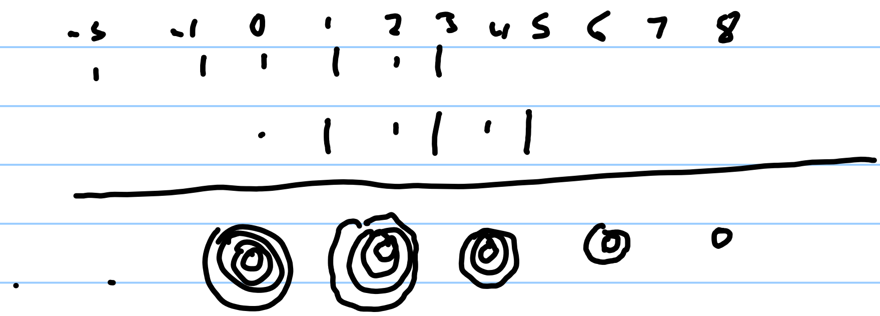
h: $-n \qquad -(n-2) \qquad\qquad +n$

... ie irreps are just $\mathrm{Sym}^n \mathbb{C}^2$.

e.g. $\mathrm{Sym}^2 \mathbb{C}^2$ = adjoint rep.

Sym^n has highest weight n, $\dim = n+1$.

Tensor product of reps: let's decompose $\mathrm{Sym}^3 \otimes \mathrm{Sym}^5$. weights = sums of weights of each:



$\Rightarrow \mathrm{Sym}^8 \oplus \mathrm{Sym}^6 \oplus \mathrm{Sym}^4 \oplus \mathrm{Sym}^2$.

.... look for highest weight vectors, four of them...

What are we doing? look at polynomial functions in x, y $\quad \mathbb{C}[x,y] = \mathbb{C}[\mathbb{A}^2]$.

Acted on by $GL_2\mathbb{C} \supset SL_2\mathbb{C}$
$$\left.\begin{array}{l} GL_2\mathbb{C} \supset SL_2\mathbb{C} \\ \quad \cup \\ \begin{pmatrix} a & \\ & a \end{pmatrix} \simeq \mathbb{C}^* \end{array}\right\} \text{commute}$$

$\Rightarrow$ decompose into eigenspaces of $\mathbb{C}^*$:

$$\mathbb{C}[x,y] = \bigoplus_{n \geq 0} \mathbb{C}[x,y]_n \quad \text{homogeneous of deg } n$$

- each of these is a rep of $SL_2\mathbb{C}$,

turns out
- each irreducible
- every rep appears exactly once!

i.e. $\mathbb{C}[x,y] = \bigoplus_{V \text{ irrep}} V$ ... like $\sqrt{\text{Peter-Weyl}}$!

Note: $\mathbb{C}^2 - 0$ is a homogeneous space for $SL_2\mathbb{C}$: $\mathbb{C}^2 \setminus 0 = SL_2\mathbb{C}/N$

$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \mathrm{Stab}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Concretely: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on x, y = basis of $(\mathbb{C}^2)^*$

$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x(v) = x\begin{pmatrix} v_1 + tv_2 \\ v_2 \end{pmatrix}$

$\xrightarrow{\left.\frac{d}{dt}\right|_{t=0}}$ $\quad e = y\frac{d}{dx}$

$SL_2\mathbb{C} \subset \mathbb{C}^4 = Mat_{2\times 2}$ so have notion of polynomial function: poly in coords $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ - which are matrix coefficients for $SL_2\mathbb{C}$

Polynomial functions on $SL_2\mathbb{C}$
$\xrightarrow{\text{restrict}}$ algebraic functions $C(SU_2)^{alg}$

... in fact an isomorphism: generated by the 2d representation, ie by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (already separates points!)

So $\mathbb{C}[SL_2\mathbb{C}] = \bigoplus_{V \text{ irred}} V \otimes V^*$

$\mathbb{C}[SL_2\mathbb{C}/N] = \bigoplus V \otimes (V^*)^N$

What are N-invariants?

Lie $N = \mathbb{C}\cdot e \Rightarrow N$ invariant means $ev = 0$.

<u>Theorem of the highest weight</u> V irrep of $SL_2\mathbb{C}$

$\Rightarrow V^N$ 1-dimensional & h-eigenspace

— every rep has a unique highest weight

$\rightsquigarrow \mathbb{C}[\mathbb{C}^2 \setminus 0] = \mathbb{C}[SL_2\mathbb{C}/N] \simeq \bigoplus_{V \text{ irrep}} V$

Interpretation via $\mathbb{CP}^1$:

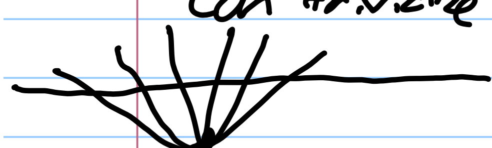
$SL_2\mathbb{C} \curvearrowright \mathbb{P}^1 = SL_2\mathbb{C}/B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ Borel subgroup

$\Rightarrow$ acts on holomorphic functions on $\mathbb{P}^1$ .... but these are just <u>constants</u>.

OTOH have a holomorphic line bundle $\begin{matrix} \mathcal{L} \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$

$\{\ell\} \in \mathbb{P}^1 \longmapsto$ fiber of $\mathcal{L}$ over $\{\ell\}$ is ℓ.

Can trivialize $\mathcal{L}$ over $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$:



canonical section

What are x,y from this POV: homogeneous coordinates on $\mathbb{P}^1$ ... don't give well defined numbers but their ratios are well defined...

$$\Longleftrightarrow \quad \begin{array}{ccc} x,y : \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ \cup & \nearrow & \\ \ell & & \end{array}$$ linear functionals on every line

$\Rightarrow$ x,y are holomorphic sections of $\mathcal{L}^* =: \mathcal{O}(1)$

$\Gamma_{hol}(\mathcal{O}(1)) = \langle x, y \rangle$

Homogeneous polynomials of deg n give sections of $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$:

fiber at ℓ are functionals $\ell \xrightarrow{f} \mathbb{C}$
$f(\lambda v) = \lambda^n f(v)$

$\mathrm{Sym}^n \mathbb{C}^2 = \Gamma_{hol}(\mathcal{O}(n))$

... find all irreps as global sections of holomorphic line bundles on $\mathbb{P}^1$: Borel-Weil

[in fact the $\mathcal{O}(n)$, $n \in \mathbb{Z}$ are all hol. line bundles on $\mathbb{P}^1$... for $n<0$ have $\Gamma(\mathcal{O}(n)) = 0$: no homog polynomials of negative degree... so in fact
irreps of $SL_2 \longleftrightarrow$ global sections of line bundles on $\mathbb{P}^1$!]

$SL_3\mathbb{C}$ / SU_3

Maximal torus 2-dimensional

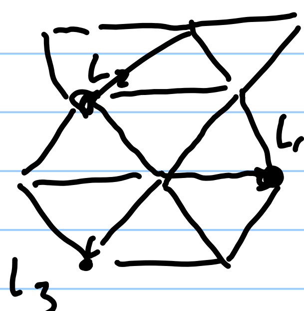
- in $\mathfrak{sl}_3\mathbb{C}$: $\mathfrak{h} = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$, $\sum a_i = 0$

$\mathfrak{h} = \mathrm{Lie}\left(T = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \quad \prod x_i = 1 \right)$

Weights : any T-rep decomposes as

$\bigoplus V_{n_1, n_2, n_3}$ with $n_1 + n_2 + n_3 = 0$

.... can draw : $\mathfrak{h}^* = \mathbb{C}\langle L_1, L_2, L_3 \rangle / \sum L_i = 0$



Weights form hexagonal lattice in $\mathfrak{h}^* \cong \mathbb{C}^2$:

__Def__ For $\lambda \in \mathfrak{h}^*$, a λ-weight vector in a rep $V \circlearrowleft \mathfrak{sl}_3$ is a vector v s.t. $h \cdot v = \lambda(h) v$ $\forall h \in \mathfrak{h}$ ... ie weights are __common eigenvalues__ for $\mathfrak{h}$.

Decomposing V under T see only __integral__ weights : live in __weight lattice__ $\Lambda \subset \mathfrak{h}^*$

= weights that vanish on

$\mathrm{Ker}(\exp : \mathfrak{h} \longrightarrow T) \simeq \mathbb{Z}^2$:

$$\begin{tikzcd}
\Lambda^\vee \arrow[dr, hook] & & \mathbb{Z} \arrow[dr, hook] & \\
& \mathfrak{h} \arrow[rr] \arrow[d, "\exp"'] & & \mathbb{R} \arrow[d, "\exp"] \\
& T \arrow[rr] & & U(1)
\end{tikzcd}$$

<u>Example</u>: adjoint rep of $\mathfrak{sl}_3$

$$\mathfrak{g} = \mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha \qquad (\mathfrak{h} = \mathfrak{g}_0)$$

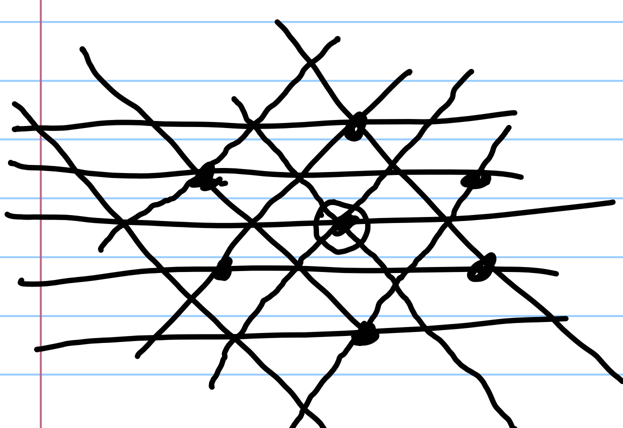
weight space decomposition

$$E_{ij} := \begin{array}{c} \begin{pmatrix} - & 1 & \\ & | & \end{pmatrix} \\ j \end{array} \quad \text{lives in } \mathfrak{g}_{L_i - L_j}:$$

$$E_{12} \quad \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}\begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix} - \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

$$= \begin{pmatrix} & a_1 & \\ & & \\ & & \end{pmatrix} - \begin{pmatrix} & a_2 & \\ & & \\ & & \end{pmatrix} = a_1 - a_2$$

So $\mathfrak{g}_\alpha = \left\{ \begin{array}{ll} \mathfrak{h} & \alpha = 0 \\ \mathbb{C}E_{ij} & \alpha = L_i - L_j \\ 0 & \text{otherwise} \end{array} \right.$ } <u>roots</u> of $\mathfrak{sl}_3$ (weights of adjoint rep)



Fundamental observation:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

.... $$[h,[x,y]] = [[h,x],y] + [x,[h,y]]$$
$$= \alpha(h)[x,y] + \beta(h)[x,y].$$

— can picture adjoint action on lattice very easily.

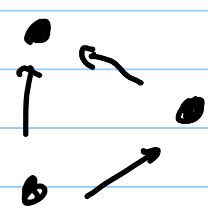
More generally, for any rep V & weight space V_α,

$$\mathfrak{g}_\alpha : V_\beta \longrightarrow V_{\alpha+\beta}$$

since $$h\cdot(x\cdot v) = x\cdot h\cdot v + [h,x]\cdot v$$
$$= \beta(h)\, x\cdot v + \alpha(h)\, x\cdot v.$$

So if we draw any rep on lattice $\Rightarrow$ picture action of root spaces ...

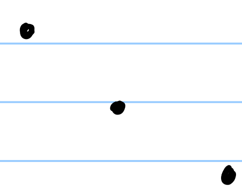
eg $\mathbb{C}^3$



etc

Note $\mathfrak{sl}_2$'s : $\begin{pmatrix} (&) \\ & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & \\ (&) \end{pmatrix}$

&



In fact have a third:

$$\Longleftrightarrow \begin{pmatrix} & 1 \\ & \end{pmatrix}, \begin{pmatrix} & \\ 1 & \end{pmatrix} \;\&\; \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\Rightarrow$ For every $\alpha \neq 0$, $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ & $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ span an $sl_2 \subset \mathfrak{g}$!

Thus we get more info on pattern of weights: in each sl_2 direction they look like sl_2-reps, ie $\oplus$ full chains in steps of 2 $\Rightarrow$

vectors in an sl_3 rep all have weights differing by integer multiples of $L_i - L_j$, not just L_i

$$\underbrace{\mathbb{Z}\langle L_i - L_j\rangle = \text{root lattice}}_{\text{lattice gen. by roots}} \subset \mathbb{Z}\langle L_i\rangle = \text{weight lattice}$$

<u>Next</u>: look for analog of <u>highest weight</u>

What does "highest" mean?

– one idea: should be simultaneously a highest weight for each of our three sl_2's:

Def A highest weight vector $v \in V$ is an $\mathfrak{h}$-eigenvector, i.e. $v \in V_\alpha$ some $\alpha \in \mathfrak{h}^*$ & annihilated by $E_{12} = \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}$, $E_{1,3} = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}$ & $E_{23} = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}$

$\Longleftrightarrow$ annihilated by $\mathfrak{n} = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ 0 & & 0 \end{pmatrix}$

$\Longleftrightarrow$ invariant under $N = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$.

Lots of ways to motivate this.

Note $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ $\quad \mathfrak{b} = \begin{pmatrix} \diagdown & * \\ & \end{pmatrix}$

$\Rightarrow$ a 1-dim rep of $\mathfrak{b}$ is trivial on $\mathfrak{n}$, factors through $\mathfrak{b}/\mathfrak{n} = \mathfrak{h}$

$\Rightarrow$ given by a weight $\alpha \in \mathfrak{h}^*$.

A 1-dim rep of B factors through $B/N = H$

$\longrightarrow$ given by <u>integral</u> $\alpha \in \mathfrak{h}^*_{\mathbb{Z}}$.

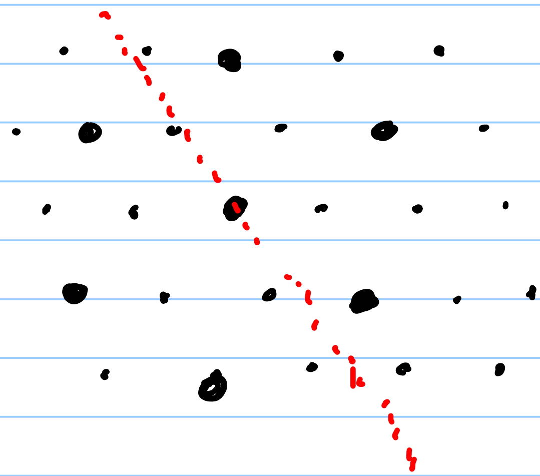
In fact <u>any</u> f.d. irrep of B is of this form ... follows from solvability of B

$\longrightarrow$ highest weight vectors are precisely the irreducible B-subreps of V....

Another perspective: to make sense of <u>highest</u>

take linear functional $l : \mathfrak{h}^*_{\mathbb{R}} \longrightarrow \mathbb{R}$

... a generic one, so no kernel of $\mathfrak{h}^*_{\mathbb{Z}} \hookrightarrow \mathbb{R}$



$\Rightarrow$ have unique weight that's highest with respect to this

$\Leftrightarrow$ invariant under $\mathfrak{g}_\alpha$ for $\alpha > 0$

(<u>positive roots</u> α)

– for appropriate l this is our $\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$. For other l we're picking different "e" in each $\mathfrak{sl}_2$ triple

$\rightsquigarrow$ get alternative notion.

<u>Proposition</u> V irrep of $\mathfrak{sl}_3$, $v \in V$ highest weight vector $\Rightarrow$ V is generated by v under applications of "lowering operators" $\underset{f_1}{(\, , \,)}, \underset{f_2}{(\, , \,)}, \underset{f_3}{(\, , \,)}$

ie $\mathfrak{n}_- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$.

Proof. Need only show e_1, e_2, e_3 preserve $U\mathfrak{n}_- \cdot v$ = Span of all words in f_1, f_2, f_3 applied to v.

Let $V_{\leq n}$ = Span of all words of length $\leq n$ in f_1, f_2 applied to v. (note $f_3 = [f_1, f_2]$)

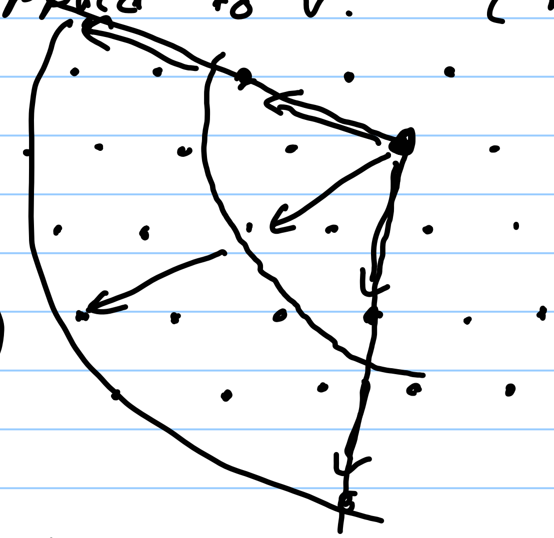
Need only 2 $\mathfrak{sl}_2$'s

Claim:

$e_i \cdot V_{\leq n} \subset V_{\leq n-1}$

$i = 1,2$

(note $e_3 = [e_1, e_2]$)



proof by induction:

$$v_n \in V_{\leq n} \Rightarrow v_n = \begin{cases} f_1 v_{n-1} & \text{or} \\ f_2 v_{n-1} \end{cases}$$

$$e_1 f_1 v_{n-1} = f_1 \underbrace{(e_1 v_{n-1})}_{\in V_{\leq n-2}} + h_1 v_{n-1} \in V_{\leq n-1}$$

$$e_1 f_2 v_{n-1} = f_2 e_1 v_{n-1}$$

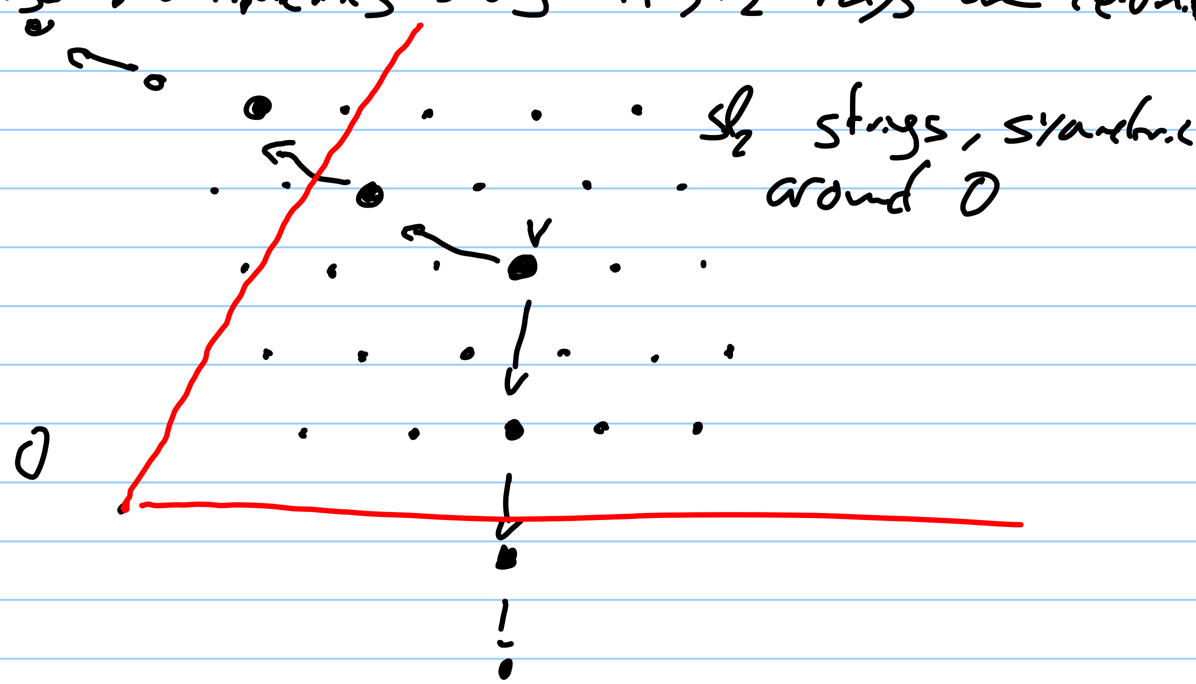
since $[e_1, f_2] = \left[\begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix}\right] = 0$

Same replacing $e_1 \longleftrightarrow e_2$. ∎

Corollary Every irrep has a <u>unique</u> highest weight vector. In fact in <u>any</u> f.d rep V, $U\mathfrak{n}_- \cdot v$ for any highest weight vector ($\mathfrak{n}_+$-invariant weight vector) is an irreducible subrepresentation
$\Rightarrow$ dim(h.w. vectors) = # irreducible summands!

Proof We see $W = U\mathfrak{n}_- \cdot v$ is $\mathfrak{g}$-invariant & has W_α f.dim. It's also irreducible since any subrep either contains $v \Rightarrow$ is everything, or has a $\mathfrak{g}$-invariant complement containing $v \Rightarrow$ is 0. ∎

Also multiplicities along f_1, f_2 rays are determined:

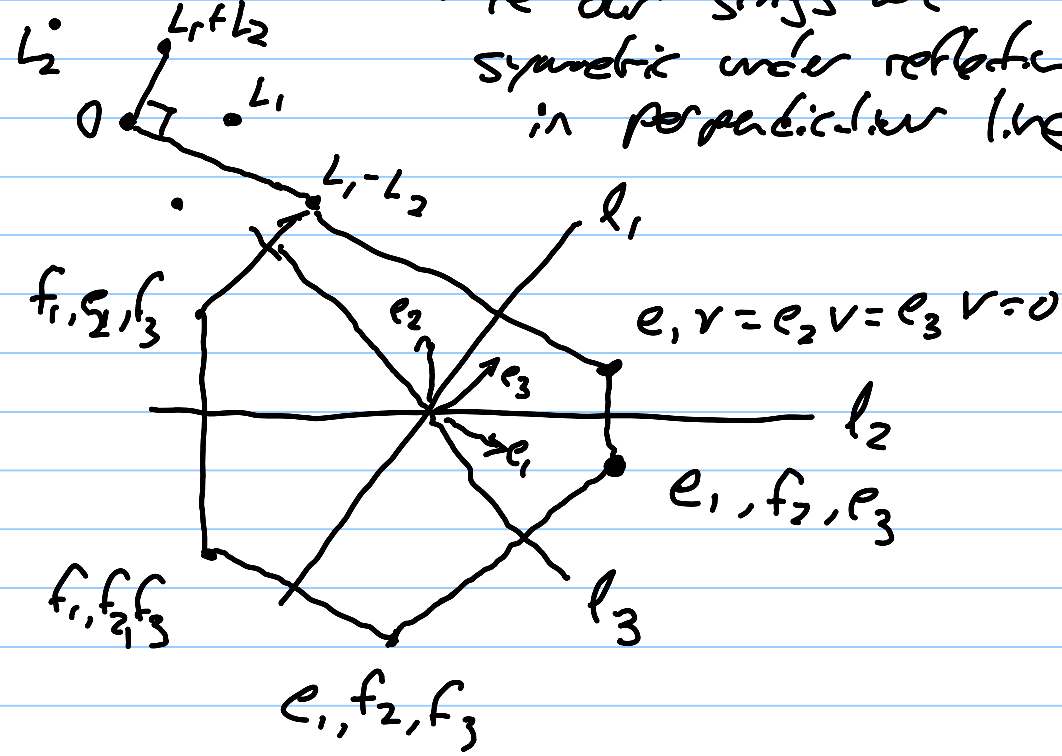


let $h_i = [e_i, f_i] \in \mathfrak{h}$
$\Rightarrow$ determine lines $l_i \in \mathfrak{h}^*$ by $\{\alpha(h_i) = 0\}$

— strings of dots are symmetric wrt the lines l_i : invariant under reflections s_i in lines l_i .

Observe : in our picture $l_i \perp \alpha_i$

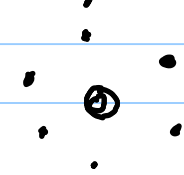
- ie our strings are symmetric under reflection in perpendicular line



— all weights of irrep lie in convex hull of $\langle s_1, s_2, s_3 \rangle \cdot \alpha$ highest weight

& now we know all the weights that appear : anything in convex hull congruent to v modulo the root lattice !

— but we don't yet know multiplicities on the inside, eg adjoint



Proposition If V, W irreps have same highest weight $\Longrightarrow V \simeq W$ (uniqueness of highest weight representations)

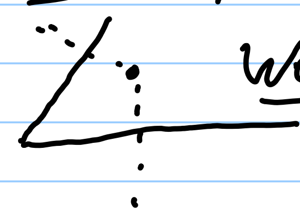
Proof Let v, w be the h.w. vectors.

$\Longrightarrow$ $v \oplus w \in V \oplus W$ is a h.w vector of same highest weight

$\Longrightarrow$ $U\mathfrak{n}_- \cdot (v \oplus w)$ irrep $\subset V \oplus W$

$\Longrightarrow$ project this onto either V or W get nonzero maps $\rightsquigarrow$ isomorphisms! $\blacksquare$

So each irrep has a label $\lambda \in \mathfrak{h}^*$

... in fact $\lambda \in$  Weyl chamber $\mathcal{C}$

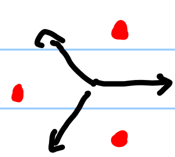
Existence: for any $\lambda \in \mathcal{C} \cap \mathfrak{h}^*_{\mathbb{Z}}$ $\exists!$ irrep V_λ.

Proof: Note $\mathbb{C}^3$ (standard) is V_{L_1}

Consider $\Lambda^2 \mathbb{C}^3$ also 3-dimensional

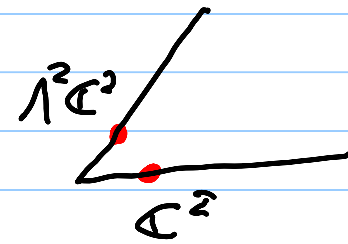
weight vectors in $\Lambda^k V$: wedges of k distinct weight vectors in V

$\Longrightarrow$ weights of $\Lambda^2 \mathbb{C}^3$ are $L_1 + L_2$, $L_1 + L_3$, $L_2 + L_3$

 $\rightsquigarrow$ distinct irrep of $\mathfrak{sl}_3$!

- these two are called the <u>fundamental reps</u>.

Note $\Lambda^0 \mathbb{C}^3 = \Lambda^3 \mathbb{C}^3 = \mathbb{C}$.



So now can construct reps

$\mathrm{Sym}^a \mathbb{C}^2 \otimes \mathrm{Sym}^b \Lambda^2 \mathbb{C}^2$

$\Rightarrow$ highest weight $aL_1 + b(L_1 + L_2)$

- not irreducible necessarily but must contain irrep $V_{(a+b)L_1 + bL_2}$.

- can find <u>any</u> integral point in Weyl chamber this way (more L_1 than L_2)

$\Leftrightarrow$ $L_1, L_1 + L_2$ are generators for weight lattice & integral generators for weights in $\mathcal{C}$, called <u>dominant weights</u> (possible highest weights).

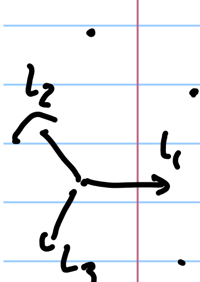
Meaning of reflections s_i: note
$\langle s_1, s_2, s_3 \rangle \cong S_3 \cong$ Weyl group $N(H)/H = W$

Explicitly each s_i is the reflection $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ in the corresponding copy $SL_2 \hookrightarrow SL_3$:
symmetry of the $\mathfrak{sl}_2$ strings in weight picture
$\mathcal{C}$ = fundamental domain for Weyl group action.

Weyl character formula for SL_3:
gives us multiplicities of highest weight rep
$V_\lambda \quad \lambda = \{(n_1, n_2, n_3) : \sum n_i = 0,\ n_1 \geq n_2 \geq n_3\}$

$$\chi_\lambda = \frac{\sum_{S_3} (-1)^i q_1^{n_1} q_2^{n_2} q_3^{n_3}}{\prod_{i<j} (q_i - q_j)} =$$

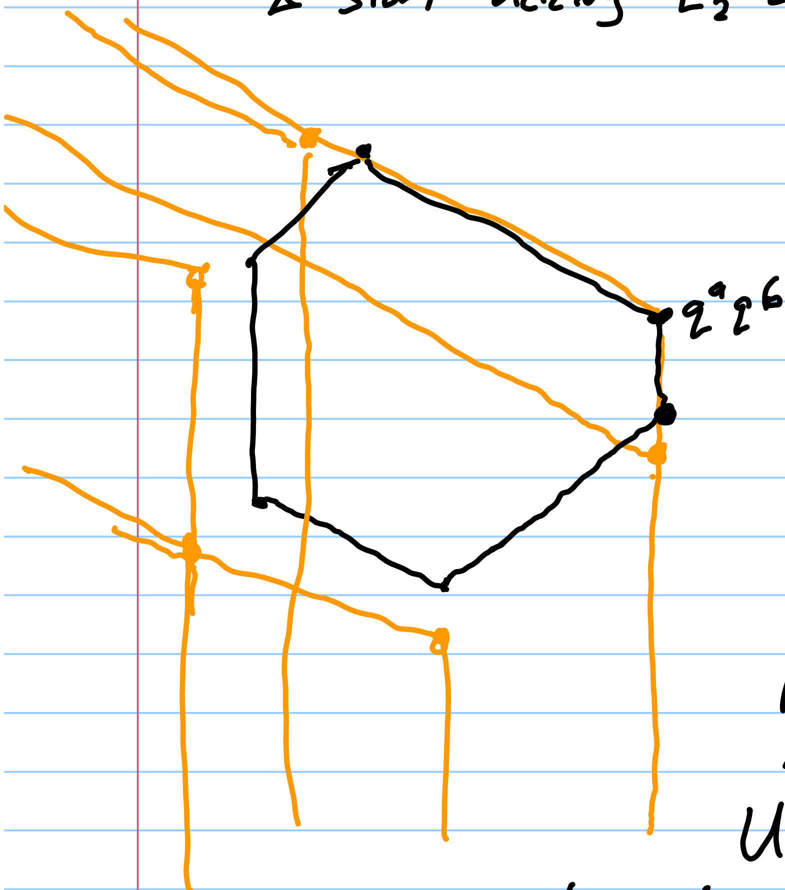
$$= \frac{q_1^{n_1} q_2^{n_2} q_3^{n_3} - \cdots\cdots}{(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)}$$



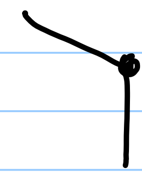
$$= \frac{q_1^{n_1} q_2^{n_2} q_3^{n_3} \quad - \cdots}{q_1^2 q_2 \left(1 - \frac{q_2}{q_1}\right)\left(1 - \frac{q_3}{q_1}\right)\left(1 - \frac{q_3}{q_2}\right)}$$

$$= q_1^a q_2^b \left(1 + \frac{q_2}{q_1} + \left(\frac{q_2}{q_1}\right)^2 + \cdots\right)\left(1 + \left(\frac{q_3}{q_1}\right) + \cdots\right)$$
$$\left(1 + \frac{q_3}{q_2} + \cdots\right) - \text{permutations}$$

each string means start with aL_1+bL_2 & start adding L_2-L_1, L_3-L_2, L_3-L_1 lowering



Get six terms all of shape



& magically cancel to give W-symmetric convex polyhedron...

looks like 6 copies of $U\mathfrak{n}_-$:

$U\mathfrak{n}_-$ is an H-rep with character $\dfrac{1}{\prod_{i<j}\left(1-\frac{q_j}{q_i}\right)}$ — as is $\mathrm{Sym}\,\mathfrak{n}_-$ (character doesn't care about algebra structure).

Two beautiful deep explanations for this:

- Atiyah-Bott localization in equivariant K-theory
- BGG resolution

Another consequence: consider V^N N-invariants. $\mathfrak{h}$ normalizes $\mathfrak{n}$ $\Rightarrow$ $\mathfrak{h}$ preserves V^N $\Rightarrow$ sum of eigenspaces.
$\rightarrow$ each eigenvector in here is actually a high weight vector.

<u>Corollary</u> V irrep $\Rightarrow$ V^N 1-dimensional
(theorem of highest weight)

$$\Rightarrow \quad \mathbb{C}[G/N] = \bigoplus_{V \text{ irrep}} V \quad !$$

each rep appears exactly once!

What is G/N?

$$\begin{array}{ccc} (\mathbb{C}^*)^2 & \longrightarrow & G/N \\ \| & & \downarrow \quad (SU_3/T) \\ H = B/N & (SL_3/B) & G/B = K/T \end{array}$$

flag manifold
- full flags in $\mathbb{C}^3$

$(\mathbb{C}^*)^2$-bundle over the flag manifold.

Can distinguish all of these reps from each other via the H action $\quad G/N \circlearrowleft H$

$$\Leftrightarrow \mathbb{C}[G/N] = \mathbb{C}[G]^N = \bigoplus V \otimes \underline{(V^*)^N}$$

sum indexed by possible highest weights $\lambda \in \Lambda \subset \mathfrak{h}^*$ which occur in irreps (in fact any $\lambda \in$ ) $\qquad$ H acts via some λ

Induced representations

$H \subset G$ finite groups, (ρ, W) rep of H

$\Rightarrow \mathrm{Ind}_H^G W$ rep of G defined as

$$\{ f \in \mathrm{Fun}(G, W) \text{ W-valued functions on } G : f(gh^{-1}) = \rho(h) \cdot f(g) \}.$$

$$= \mathrm{Map}_H(G, W)$$

G acts via left translation:

$$g \cdot f(k) = f(gk)$$

– value on each H coset determined by definition $\rightsquigarrow$ as vector space

$$\mathrm{Ind}_H^G W \simeq \bigoplus_{[g] \in G/H} [g] \cdot W$$

– vector bundle over G/H with fibers $\simeq W$.

Can find this as an <u>associated vector bundle</u>:

$$\begin{array}{c} H \hookrightarrow G \\ \downarrow \\ G/H \end{array}$$

fibers are H-cosets – replace by copies of W:

$$G \times_H W = \{ (g, w) \in G \times W \} / (g, w) \sim (gh^{-1}, hw)$$

$\longleftrightarrow (g, hw) \sim (gh, w)$: "tensor over H"

$$\begin{array}{c} G\times_H W \\ \downarrow \\ G/H \end{array}$$ is a vector bundle with fibers $\simeq W$.

Sections of $G\times_H W \longleftrightarrow \mathrm{Ind}_H^G W$.

— $G \times_H W$ is a G-equivariant vector bundle:

for every $g \in G$ get isomorphism

$$g : G\times_H W|_{g_1} \longrightarrow G\times_H W|_{gg_1}$$

$\rightsquigarrow$ sections form a G-representation

(pulling back a section gives a section of the same bundle.... in particular fibers are reps of stabilizer groups)

Categorical meaning of induction:

left adjoint to restriction:

$W \in \mathrm{Rep}\, H$, $V \in \mathrm{Rep}\, G$

$$\Rightarrow \quad \mathrm{Hom}_{\mathrm{Rep}\,H}(W, \mathrm{Res}_H^G V) = \mathrm{Hom}_{\mathrm{Rep}\,G}(\mathrm{Ind}_H^G W, V)$$

Frobenius reciprocity

... G rep freely generated by H-rep W:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \text{ any } G\text{-rep} \\ \mathrm{Ind}_H^G W \arrow[ur, dotted] & \end{tikzcd}$$

$$\longleftrightarrow \boxed{\mathrm{Ind}_H^G W = \mathbb{C}G \otimes_{\mathbb{C}H} W}$$

Case of Lie groups: still get vector bundle over G/H, but now lots of different notions of sections (smooth, ...)

- ie $Map_H(G,W)$ can mean smooth, holomorphic, ... different kinds of induction...
  though to get fin dim reps need strong restrictions on type of map & subgroup

Our case: $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$.

G/B compact complex manifold $\Rightarrow$ have strong finite dimensionality for holomorphic sections of vector bundles...

W will be 1-dim rep of $B \longleftrightarrow$ character $\exp\lambda : H \to \mathbb{C}^*$ - defines G-equivariant line bundle $\mathcal{O}(-\lambda)$ on B by $G \times_B \mathbb{C}_\lambda$

Construction V irrep with highest weight λ, vector v

$$\Rightarrow \text{map } V^* \longrightarrow Map_B^{hol}(G_{\mathbb{C}}, \mathbb{C}_{-\lambda})$$
$$\alpha \longmapsto \{ g \longmapsto \alpha(gv) \}$$

- ie look at orbit of v, $G \longrightarrow V$, $g \longmapsto v$

& restrict α to a holomorphic function on V

Borel-Weil This is an isomorphism, ie
$$V^* \simeq \mathrm{Ind}_B^G \mathbb{C}_{-\lambda} = \Gamma(G/B, \mathcal{O}(\lambda))$$

Proof Map is injective since $\neq 0$ & V^* irreducible
So need to see RHS irreducible.
Suffices to see that Ind has at most one lowest weight vector, ie vector fixed by $N_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$.

If $f \in \mathrm{Map}_B(G, \mathbb{C})$ fixed by N_-
$\Rightarrow f(n_- b) = e^{-\lambda}(b) \cdot f(1) \Rightarrow$
$f|_{N_- B \subset G}$ completely determined by $f(1)$.

But $N_- B \subset G$ is open (& dense):
$T_{[B]} G/B \simeq \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}_-$, so
$N_- \longrightarrow G/B$ has differential which is an isomorphism so image open (in fact dense.... later) but hol. fns on a connected space agreeing on open set must agree $\Longrightarrow$ 1-dimensional! ∎

This also proves generically that $V^N \simeq \mathbb{C}$ for V irreducible - once we know reps have highest & lowest weight vectors: comes from openness of $N_- B \subset G$!

Interpretation via projective geometry:

$v \in V$ highest weight vector $\Rightarrow$ B stabilizes line $\mathbb{C}v$

So get orbit map

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & \mathbb{P}V \\ & G/B \arrow[ur] & \end{tikzcd}$$

On $\mathbb{P}V$ have line bundle $\mathcal{O}(1)$, dual to tautological line bundle — restricts to $\mathcal{O}(\lambda)$ on G/B (hence $-\lambda$ in definition), sections are exactly V^* (both on $\mathbb{P}V$ & on G/B — no linear functionals can vanish on orbit of v, by irreducibility!)

<u>Bruhat decomposition</u> $\longleftrightarrow$ Gaussian elimination:

Study column operations on a basis $g = \{v_1 \dots v_n\}$ of $\mathbb{C}^n$ — multiply by a scalar or subtract multiples of vectors to left .... ie find basis of flag, gB.

Reduced echelon form: can put in form

$$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

: each column ends with a 1, entries to its right vanish

$\Rightarrow$ after permutation land in $N = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$\Rightarrow$ Any $g = n\pi b \quad n \in N \ \pi \in S_n \ b \in B.$

$\rightsquigarrow \quad N \backslash G / B \longleftrightarrow S_n$ $\quad$ N-orbits of flags labelled by permutations

$B \backslash G / B$

Can be more precise: to make factorization unique: given g get permutation π, & row echelon w means that

- $w\pi^{-1} \in N$ $\quad$ (defines π from row echelon w)
- $\pi^{-1} w \in N_-$

$$\begin{pmatrix} * & 1 & & \\ * & & * & 1 \\ 1 & & & \\ & & 1 & \end{pmatrix} \begin{pmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{pmatrix} = \begin{pmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{pmatrix} \begin{pmatrix} * & 1 & & \\ * & & * & 1 \\ 1 & & & \\ & & 1 & \end{pmatrix} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ & & 1 & \\ * & & * & 1 \end{pmatrix}$$

$\Rightarrow$ $g = n\pi b$ <u>uniquely</u> with $b \in B, \pi \in S_n$ & $n \in N \cap \pi N_- \pi^{-1}$

$$\Leftrightarrow \quad G/B = \coprod_{S_n} N \cap \pi N_- \pi^{-1}$$

Dimension of orbit = length of π = # of <u>crossings</u> $i<j$ but $\pi_j > \pi_i$

= length of 2-cycle description of π.

<u>Corollary</u> $G/B \simeq \coprod_{\pi \in S_n} \mathbb{C}^{\ell(\pi)}$ <u>$n!$ cells</u>

<u>Corollary</u> $H^i(G/B, \mathbb{Z}) \simeq \bigoplus_{\substack{\pi \in S_n \\ \ell(\pi) = i}} \mathbb{Z}$.

In particular have an <u>open</u> <u>orbit</u>

$$N_- \subset G/B \quad , \dim N_- = \dim G/B = \tfrac{1}{2} n(n-1)$$

ie almost all $g \in G$ have a unique factorization

$$g = n_- b.$$

Its existence $\longleftrightarrow$ uniqueness of highest weight

More detailed analysis shows $n!$ orbits in natural bijection with $n!$ geometric series in Weyl character formula . . .

<u>Meta-theorem</u> Geometry of $G/B \cong$ Representation Theory of G