Lie Groups, Part 4

Note Title

11/16/2009

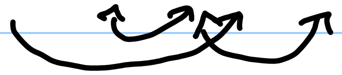
Schur-Weyl duality

V vector space, $V^{\otimes 2} = \mathrm{Sym}^2 V \oplus \Lambda^2 V$.

$V^{\otimes 3} = \mathrm{Sym}^3 V \oplus \Lambda^3 V \oplus$ something.... how to describe?

Idea: $V^{\otimes k}$ carries two natural group actions:

- GL_n $\quad n = \dim V$
- S_k symmetric group permuting factors $\quad V \otimes V \otimes \dots \otimes V$



These actions commute! ⇒ action of $GL_n \times S_k$. So if we decompose

$$V^{\otimes k} = \bigoplus_{\substack{W \text{ irrep} \\ \text{of } S_k}} W \otimes V_W^{\otimes k}$$

$V_W^{\otimes k}$ ← multiplicity space $= \mathrm{Hom}_{S_k}(W, V^{\otimes k})$

⇒ each multiplicity $V_W^{\otimes k}$ is a rep. of GL_n.

Conversely $V^{\otimes k} = \bigoplus_{\substack{U \text{ irrep} \\ \text{of } GL_n}} U \otimes V_U^{\otimes k}$ ← rep of S_k

Theorem! $V_W^{\otimes k}$ are irreducible reps of GL_n

$V_U^{\otimes k}$ " " " " S_k

$\Rightarrow$ bijection between irreps of GL_n & S_k which appear in $V^{\otimes k} = \underset{\substack{\lambda \\ \text{some labels}}}{\bigoplus} W_\lambda \otimes U_\lambda$

2. Moreover <u>all</u> irreps of S_k appear this way for $n \geq k$ - so λ labels irreps of S_k (ie $\lambda \vdash k$ partition) & all irreps of GL_n appear this way for k large enough (up to powers of the determinant rep).

<u>Proof</u> 2: First if $n \geq k$, e_i basis of V $\Rightarrow$ find free S_k orbit by taking k distinct e_i & acting on $e_{i_1} \otimes \dots \otimes e_{i_k}$ $\Longrightarrow$ spans a copy of $\mathbb{C}S_k \subset V^{\otimes k}$, but every irrep of S_k appears in there.

Note matrix elements of all subreps of all $V^{\otimes k}$ form a subring of $\mathbb{C}[GL_n]$ .. but $\mathbb{C}[GL_n]$ is generated by matrix elements a_{ij} of V itself together with $\frac{1}{\Delta}$, Δ = determinant !

1. Study endomorphisms :

$$\mathrm{End}_{S_k} V^{\otimes k} = \bigoplus_W \mathrm{End}\, V_W^{\otimes k}$$

$$\mathrm{End}_{G \times S_k} V^{\otimes k} = \bigoplus_W \mathrm{End}_G\, V_W^{\otimes k}$$

→ want to show each of these is just $\mathbb{C}$.

So now we're really trying to prove $V^{\otimes k}$ is *multiplicity-free* as a $GL_n \times S_k$ rep $\iff$ its endomorphisms are *commutative* (no matrix algebras!)

- ie suffices to show

$$\mathrm{End}_{GL_n \times S_k} V^{\otimes k} \subset \mathrm{Center}(\mathrm{End}_{S_k} V^{\otimes k})$$

hence to show $\mathrm{Im}(GL_n) \subset \mathrm{End}_{S_k} V^{\otimes k}$ *spans* it as a vector space (so commuting with GL_n, here with its image, means commuting with everything).

But $\mathrm{End}_{S_k}(V^{\otimes k}) = \left(V^{\otimes k} \otimes V^{*\otimes k}\right)^{S_k}$

$$= \left((\mathrm{End}\, V)^{\otimes k}\right)^{S_k} = \mathrm{Sym}^k\, \mathrm{End}\, V$$

So linear forms on this vector space are degree k polynomial on $(\mathrm{End}\, V)^* = \mathrm{End}\, V$ & $GL_n \subset \mathrm{End}\, V$ is dense so no polynomials vanish on it $\Rightarrow$ GL_n spans $\mathrm{End}_{S_k} V^{\otimes k}$.

∎

Schur functors fix W_λ irrep of S_k $\Rightarrow$ functor $V \longmapsto \mathbb{S}_\lambda V = \lambda$-component of $V^{\otimes k}$

eg λ : trivial $\Rightarrow$ $\mathbb{S}_{triv} V = Sym^k V$

λ = sign $\Rightarrow$ $\mathbb{S}_{sign} V = \Lambda^k V$

S_3 : three irreps : also $\mathbb{C}^2 = \{x+y+z=0\} \subset \mathbb{C}^3 \circlearrowleft_{S^3}$

$(1+1+2^2 = 6 = |S_3|)$

$$\Rightarrow V^{\otimes 3} = Sym^3 V \oplus \Lambda^3 V \oplus \underbrace{\mathbb{C}^2 \otimes Hom_{S_3}(\mathbb{C}^2, V^{\otimes 3})}_{\{\sum a_{ijk} e_i \otimes e_j \otimes e_k \; : \; a_{ijk} + a_{jki} + a_{kij} = 0\}}$$

- distinguish two summands by $a_{ijk} = \pm a_{jik}$
  - ie eigenspaces of (12) on $\mathbb{C}^2$.

__Def__ $G \curvearrowright V \curvearrowleft H$ are said to be a dual pair if their spans in End V are each other's centralizers.

__Corollary of proof__ $G \curvearrowright V \curvearrowleft H$ dual pair

$$\Rightarrow V = \bigoplus \underset{\text{irrep of } G}{W_i} \otimes \underset{\text{irrep of } H}{U_i}$$

$\Rightarrow$ bijection between irreps of G, H which happen to appear in V.

eg $G \curvearrowright \mathbb{C}[G] \curvearrowleft G$ or $G \curvearrowright \mathbb{C}[G/N] \curvearrowleft H$

- more exciting when find genuinely different groups!
  - key construction behind __Langlands program__

(roughly $G = Gal\, \bar{\mathbb{Q}}/\mathbb{Q}$ $\quad H = \prod_p Gl_n \mathbb{Q}_p \times Gl_n \mathbb{R} \ldots$)

Structure theory

Def Lower central series of a Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} \supset D_1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \supset D_2\mathfrak{g} = [\mathfrak{g}, D_1\mathfrak{g}] \supset$$
$$\cdots \supset D_k\mathfrak{g} = [\mathfrak{g}, D_k\mathfrak{g}]$$

- all in fact ideals in $\mathfrak{g}$

Upper central series

$$\mathfrak{g} \supset D^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \supset \ldots \supset D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}] \supset ..$$

all (just) subalgebras

Def $\mathfrak{g}$ is nilpotent if $D_k\mathfrak{g} = 0$ some k

solvable if $D^k\mathfrak{g} = 0$ some k

Example $\mathfrak{n} = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $[\mathfrak{n},\mathfrak{n}] = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix}$

$D_k\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 & * \\ & \ddots & \ddots & \\ & & & \end{pmatrix} \Rightarrow \mathfrak{n}$ nilpotent

$\mathfrak{b} = \begin{pmatrix} \times & & * \\ & \ddots & \\ 0 & & \times \end{pmatrix}$ $\quad [\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$, $[\mathfrak{n},\mathfrak{n}] = \ldots$

$\Rightarrow \mathfrak{b}$ solvable. BUT

$[\mathfrak{b},\mathfrak{n}] = \mathfrak{n}$ so not nilpotent.

Def $\mathfrak{g}$ is simple if $\dim \mathfrak{g} > 1$ &

$\mathfrak{g}$ contains no nontrivial ideals $[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}$..

$\mathfrak{g}$ is semisimple if $\mathfrak{g}$ has no nonzero solvable ideal's

$\mathfrak{g}$ solvable $\Longleftrightarrow$ has sequence $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_t = 0$ of subalgebras s.t. $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ abelian (or even 1-dim).

So if $\mathfrak{g} \supset \mathfrak{h}$ ideal, $\mathfrak{g}$ solvable $\Longleftrightarrow$ $\mathfrak{h}$ solvable & $\mathfrak{g}/\mathfrak{h}$ solvable

$\Longrightarrow$ sum of solvable ideals is a solvable ideal (since $\mathfrak{a}+\mathfrak{b}/\mathfrak{b} = \mathfrak{a}/\mathfrak{a}\cap\mathfrak{b} \dots$)

$\Longrightarrow$ have a <u>maximal</u> solvable ideal in any $\mathfrak{g}$:

$$0 \to \underset{\underline{\text{radical}}}{\mathrm{Rad}\,\mathfrak{g}} \to \mathfrak{g} \to \underset{\underline{\text{semisimple}}}{\mathfrak{g}/\mathrm{Rad}\,\mathfrak{g}} \to 0$$

<u>Prop</u> The adjoint rep of $\mathfrak{g}$ is faithful iff $\mathfrak{g}$ is semisimple

<u>Proof</u> $\mathrm{Rad}\,\mathfrak{g} \neq 0 \Rightarrow$ find abelian ideal $\Rightarrow$ center $\neq 0$.
Conversely $\mathfrak{g}$ semisimple $\Rightarrow Z(\mathfrak{g}) = 0 \Rightarrow$ ad faithful! ∎

<u>Engel's Theorem</u> Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a subalgebra s.t. every $x \in \mathfrak{g}$ is a nilpotent endomorphism of V
$\Longrightarrow \exists\, v \in V$ s.t. $x \cdot v = 0 \quad \forall x \in \mathfrak{g}$.

<u>Corollary</u> $\mathfrak{g}$ preserves a full flag, hence is conjugate under $GL\,V$ into $\mathfrak{b} = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$.

Pf of corollary by induction: look at $V/\mathbb{C}v$ & apply Engel again.

Proof of Theorem: First note x nilpotent on V $\Rightarrow$ ad x nilpotent on $\mathfrak{gl}\,V$:

x nilpotent $\rightsquigarrow$ have a flag $0 \subset V_1 \subset V_2 \subset \dots \subset V$ with $xV_i \subset V_{i-1}$. So for any $y \in \mathfrak{g}$ $(\operatorname{ad} x)^k y$ takes V to smaller & smaller subspaces.

Induction on dim $\mathfrak{g}$.

Claim $\mathfrak{g}$ contains $\mathfrak{h}$ ideal of codim one: let $\mathfrak{h}$ be a maximal proper subalgebra

Any $x \in \mathfrak{h}$ acts nilpotently on V $\rightsquigarrow$ hence on $\mathfrak{g} \subset \mathfrak{gl}(V)$ $\rightsquigarrow$ hence on $\mathfrak{g}/\mathfrak{h}$

By induction $\exists\, \bar{y} \in \mathfrak{g}/\mathfrak{h}$ killed by all $\mathfrak{h}$

$\rightarrow$ adding y to $\mathfrak{h}$ we get a subalgebra $\Rightarrow$ codim $\mathfrak{h}=1$ & $\mathfrak{h}$ is an ideal!

Induction hypothesis $\Rightarrow$ $\mathfrak{h}$ kills some $v \in V$. Let $W = V^{\mathfrak{h}}$ (all v s.t. $\mathfrak{h}\cdot v = 0$)

let $y \in \mathfrak{g}$ not in $\mathfrak{h}$.

$$\forall\, w \in W\ x \in \mathfrak{h} \qquad x(yw) = \underset{=0}{y(\overset{0}{\overset{\shortparallel}{xw}})} + \overset{\in \mathfrak{h}}{[x,y]}\,w$$

$\Rightarrow yv \in W$ ie $y : W \to W$, nilpotent

$\Rightarrow \exists\, w \in W$ with $yw = \lambda w = 0 \Rightarrow \mathfrak{g} w = 0$

$\blacksquare$

Lie's Theorem $\mathfrak{g} \subset \mathfrak{gl}\, V$ solvable

$\Rightarrow \exists$ nonzero eigenvector for $\mathfrak{g}$ in V

Corollary $\mathfrak{g}$ is conjugate to a subalgebra of $\mathfrak{b}$, ie $\mathfrak{b} \subset \mathfrak{gl}_n$ is a maximal solvable subalgebra & any two such are conjugate.

[analog of p-Sylow subgroups: check $B \subset GL_n \mathbb{F}_p$ is a p-Sylow...]

Lemma $\mathfrak{h} \subset \mathfrak{g}$ an ideal in any Lie algebra, V a rep of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$

$$V_\lambda = \{ v \in V : h \cdot v = \lambda(h) v \quad \forall h \in \mathfrak{h} \}$$

is preserved by $\mathfrak{g}$

[Note: λ is not assumed a Lie algebra map $\mathfrak{h} \to \mathbb{C}$ so not obvious by normality]

Proof Let $0 \neq w \in W$, $x \in \mathfrak{h}$, $y \in \mathfrak{g}$

$$(*) \quad xyw = yxw + [x,y]w$$
$$= \underset{\in W}{\lambda(x)\, yw} + \underset{??}{\lambda([x,y])}\, w$$

need $\lambda([x,y])=0$ all $x\in\mathfrak{h}$

[– in particular $\lambda([\mathfrak{h},\mathfrak{h}])=0$ &
λ $\underline{\underline{\text{is}}}$ a homomorphism $\mathfrak{h}\to\mathbb{C}$!]

Let $U=\text{Span}\{w, yw, y^2w, \dots\}\subset V$

Claim: $x\in\mathfrak{h}$ preserves U: acts upper triangularly on above basis in fact by (*) above & all diagonal entries given by $\lambda(x)$.

$\Rightarrow$ trace is $\lambda(x)\dim U$.

Apply to $x'=[x,y]$: trace $=\lambda([x,y])\dim U$

but trace of commutator $=0$ $\Rightarrow$ $\lambda([x,y])=0$ ■

Proof of Lie's Theorem

$\mathfrak{g}$ solvable $\Rightarrow \mathfrak{a}:=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\neq 0$

$\Rightarrow$ inverse image of codim 1 subspace of $\mathfrak{a}$ is codim one ideal in $\mathfrak{g}$, call it $\mathfrak{h}$.

Induction $\Rightarrow$ $\exists\, v_0\in V$ eigenvector for $\mathfrak{h}$, eigenvalue $\lambda\in\mathfrak{h}^*$. Let $W=V_\lambda$.

Lemma $\Rightarrow$ for $y\in\mathfrak{g}\setminus\mathfrak{h}$, $y\cdot W\subset W$

$\Rightarrow$ has eigenvector in W ✓ ■

Corollary Any finite dimensional irrep of a solvable Lie algebra is 1-dimensional

Pf Image $\mathfrak{g} \longrightarrow \mathfrak{gl}\,V$ is a quotient of $\mathfrak{g}$ by an ideal, hence solvable, hence preserves a flag. $\blacksquare$

The Heisenberg algebra

L vector space (eg $\mathbb{R}$)

$V = T^*L = L \oplus L^*$ $(\mathbb{R}^2)$ has canonical symplectic (nondegenerate skew) form where $\langle v \oplus w, v' \oplus w' \rangle = w'(v) - w(v')$

$\Rightarrow$ Construct Lie algebra $0 \to \mathbb{R}1 \to \mathfrak{h} \to V \to 0$ as a <u>central</u> <u>extension</u> of $(V, [,]=0)$: ie 1 is central. As a vector space write $\mathfrak{h} = V \oplus \mathbb{R}1$. Lie bracket

from $\left\{ \begin{matrix} V \times V \longrightarrow V \\ \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ V \times \mathbb{R} \longrightarrow \mathfrak{h} \end{matrix} \right\}$ is given (by zero),

$\rightsquigarrow$ defined completely by a skew form $V \times V \longrightarrow \mathbb{R}$: use $\langle , \rangle$!

ie $[v \oplus c1, v' \oplus c'1] = \langle v, v' \rangle \cdot 1$.

Case $L = \mathbb{R}$: $\mathfrak{h} \simeq \begin{pmatrix} 0 & \bullet & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}$

In general $\mathfrak{h}$ is <u>2-step nilpotent</u>, ie $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = 0$.

So in any f.d irrep of $\mathfrak{h}$, $\underline{1}$ acts by zero.

But have an interesting <u>∞-dim</u> irrep of $\mathfrak{h}$.

Let $\mathcal{H} = \mathrm{Sym}\, L^* \simeq$ polynomial functions on L.

$\mathfrak{h}$ acts on $\mathcal{H}$ as follows:

- $L^* \subset V \subset \mathfrak{h}$ acts by <u>multiplication</u> by $Q_j = i x_j$
- $L \subset V \subset \mathfrak{h}$ acts by <u>differentiation</u> $P_j = \frac{\partial}{\partial x_j}$
- $[P, Q] = -i$ $\quad ([\partial, x] = 1)$
- $\underline{1}$ acts by 1.

In fact $U\mathfrak{h} \simeq \mathcal{D}_L = \mathbb{R}\langle x_i, \partial_i \rangle$ algebra of polynomial differential operators on L acts on $\mathcal{H} = \mathbb{C}[L]$.

$\mathcal{H}$ is irreducible: from any polynomial can get to a constant by differentiation

... $L \cdot 1 = 0$, analog of highest weight vector.

$\mathcal{H} \simeq UL^* \cdot 1 \simeq \mathrm{Sym}\, L^*$ freely generated by 1.

Another irrep $\mathcal{H}^\vee$: all distributions on L supported at $0 \in L$: act by multiplication & differentiation. Start with vector δ with $L^* \cdot \delta = 0$ (ie $x_i \cdot \delta = 0 \ \forall i$) & generate freely under L, $\mathcal{H}^\vee \cong \mathrm{Sym}\, L \cdot \delta$ all derivatives of δ. Again can bring any vector in $\mathcal{H}^\vee$ to δ by L^* operations $\Rightarrow$ irreducible.

Can describe $\mathcal{H} = \mathcal{D} \cdot 1$, $\mathcal{H}^\vee = \mathcal{D} \cdot \delta$

Case $L = \mathbb{R}$: another irrep is given by $\mathcal{D} \cdot e^{-\frac{1}{2}x^2}$: all polynomial diffops $\mathcal{D} = \mathbb{C}\langle x, \partial \rangle$ applied to $e^{-\frac{1}{2}x^2} = \Omega$

$a^* = \frac{1}{\sqrt{2}}(P - iQ) = -\frac{i}{\sqrt{2}}(\partial + x)$ annihilates Ω generate freely under $a = \frac{1}{\sqrt{2}}(P + iQ) = \frac{1}{\sqrt{2}}(\partial - x)$, & $\boxed{[a^*, a] = 1}$ (creation & annihilation)

More generally can take any Lagrangian subspace $L' \subset V$: maximal subspace on which $\langle , \rangle$ vanishes (isotropic). ($\Rightarrow V/L \cong L^*$).

Construct irrep $\mathcal{H}_{L'} = \mathcal{U}h \cdot v$,
v vector with $L' \cdot v = 0$ & $\mathbb{1} v = v$.

Can't have anything outside L' annihilate v since $\mathbb{1} v \neq 0$

- in fact $\mathcal{H}_{L'} \simeq \mathrm{Sym}\, V/L'$.

All of these irreps are related by *outer automorphisms* of $\mathfrak{h}$: $Sp(V)$ symplectic group of V acts, takes Lagrangians to Lagrangians & identifies all the different vector spaces $\mathcal{H}_L$.

Focus on special case $V = \mathbb{R}^2$

[$Sp(V) = SL_2\mathbb{R}$: skew forms on V are given by $\Lambda^2 V^*$ - so preserving a skew form is equivalent to having determinant one.]

Let's consider the group analog of these reps.

H = Heisenberg group $1 \to \mathbb{T} \to H \to \mathbb{R}^2 \to 0$ group central extension: as manifold $H = \mathbb{R}^2 \times \mathbb{T}$,

$$u \exp \xi \cdot v \exp \eta = uv\, e^{i\langle \xi, \eta \rangle} \exp(\xi + \eta)$$

H acts on $L^2(\mathbb{R})$ via

$$T_a f = f(x-a) \qquad M_b f = e^{2\pi i b x} f(x)$$

$$U_c f = e^{2\pi i c} f;$$ on *smooth functions*

These exponentiate differentiation, multiplication & scalar operators, but $\mathfrak{h}$ doesn't act on L^2 (well - as unbounded operators...)

Same for n-dimensional Heisenberg group

$$1 \to \mathbb{T} \to H \to \mathbb{R}^{2n} \to 0$$

acting on $L^2(\mathbb{R}^n)$.

(*) $L^2(\mathbb{R}^n)$ is irreducible : no closed invariant subspace (we saw this on Lie algebra level .... $\rightsquigarrow$ on group, modulo some basics of ∞-dim rep theory)

Let $\wedge : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ be the Fourier transform $\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot y} dx$

$\wedge$ intertwines x & ∂_x (up to sign)
$\wedge$ " T_a & M_a (" " ")

Since L^2 is irreducible $\Longrightarrow$ this implies $\wedge$ is unitary up to a scalar — & can find this scalar by calculating <u>any single</u> Fourier transform

- eg $(e^{-\frac{x^2}{2}})^\wedge = e^{-\frac{x^2}{2}}$: again know it's true up to scalar since its diff eqn $a^* \Omega = 0$ is Fourier invariant (once we put in i's in correct places) & check norm of $e^{-\frac{x^2}{2}}$ is one .....

$\Longrightarrow$ Plancherel theorem, $\wedge$ is an isometry.

Compare with Lie algebra: $\mathcal{U}_c$'s all related by outer automorphisms of $\mathfrak{h}$. Passing to Lie groups though we find the same *vector space*: only one representation!

Stone-von Neumann Theorem Any unitary irrep of H on which T acts as scalars is isomorphic to $L^2(\mathbb{R}^n)$! strong uniqueness.

"Oscillator model": while 1 & δ_0 not in L^2, $\Omega = e^{-\frac{1}{2}x^2}$ is ..
find $\mathbb{C}[a]\cdot\Omega \subset L^2\mathbb{R}$ dense subspace ... quantum mechanics of simple harmonic oscillator. . a^* acts as $\frac{d}{da}$.

Ω looks like an sl_2 highest weight vector!
Recall $Sp(V)$ acts on $\mathfrak{h}$ or H, hence on set of reps of each.
But H has only one irrep! so for each $g \in Sp(V)$ can find $T_g : L^2\mathbb{R}^n \circlearrowleft$ unitary such that for $h \in H$,

$$T_g \circ h = (g\circ h)\circ T_g .$$

Each unique up to scalar $\rightsquigarrow$ define a *projective representation* of $Sp(V)$:

action of $Sp(V) \curvearrowright \mathbb{P}(L^2(\mathbb{R}^n))$.

eg. $n=1$ $\quad SL_2\mathbb{R} \curvearrowright \mathbb{P}(L^2(\mathbb{R}))$.

for example $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts as Fourier transform!

$\rightsquigarrow$ these operators are generalizations . . .

Turns out we can't choose T_g's consistently to define an action of $SL_2\mathbb{R}$ – but can on level of Lie algebra:

consider $\left\{ \frac{i}{2}P^2, \frac{i}{2}(PQ+QP), \frac{i}{2}Q^2 \right\} =$

$$\left\{ -\frac{i}{2}\partial^2, x\partial + \frac{1}{2}, \frac{i}{2}x^2 \right\}$$

– not vector fields on $\mathbb{R}$! but satisfy relations of $\mathfrak{sl}_2\mathbb{R}$:

eg $\left[x\partial + \frac{1}{2}, \frac{i}{2}x^2 \right] = 2\frac{i}{2}x^2$

$$[\partial^2, x^2] = \partial\partial xx - xx\partial\partial$$
$$= \partial x\partial x + \partial x - xx\partial\partial$$
$$= x\partial\partial x + 2\partial x - xx\partial\partial$$
$$= x\partial x\partial + x\partial + 2\partial x - xx\partial\partial = 2x\partial + 2\partial x = 4x\partial + 2$$

Relation to $SL_2\mathbb{R}$: let's focus on

$T \cong SO_2 \subset SL_2\mathbb{R}$:

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Corresponding Lie algebra generator: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- conjugate in $SL_2\mathbb{C}$ to $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$

or in $sl_2\mathbb{C}$ $\rightsquigarrow$ $\begin{pmatrix} i & \\ & -i \end{pmatrix} = ih$.

So h acts by $\frac{1}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right) = aa^* + \frac{1}{2}$

$$= \frac{1}{2}(P^2 + Q^2)$$

The "h" element is the Hamiltonian/energy of the harmonic oscillator, Ω is the vacuum vector – lowest eigenstate

- acts as $a\partial_a + \frac{1}{2}$, eigenvectors $a^n\Omega$ with spectrum $\{n + \frac{1}{2}\}$.

These are not integers! so $\mathbb{T} \simeq SO_2 < SL_2\mathbb{R}$ doesn't act – but its double cover does!

Recall $\pi_1 SL_2\mathbb{R} = \mathbb{Z}$ $\quad (SL_2\mathbb{R} \simeq D^2 \times \mathbb{T})$

$\Rightarrow \exists!$ double cover $\mathbb{Z}/2 \to Mp_2 \to SL_2\mathbb{R}$

the <u>metaplectic group</u>. $[\mathbb{Z}/2 \to Mp_n \to Sp_n\mathbb{R}]$

So $Mp_n \circlearrowright L^2\mathbb{R}^n$ oscillator/Weil/metaplectic representation

- includes Fourier transform & lots of other great operators!

__Prop__ Mp_2 (& all other covering groups of $SL_2\mathbb{R}$, eg $\widetilde{SL_2\mathbb{R}}$) are not matrix groups

__Proof__ We know any fin dim rep of their Lie algebra $\cong \mathfrak{sl}_2\mathbb{R}$ is actually a rep ... of $SL_2\mathbb{R}$ so covering group acts trivially! $\boxtimes$

[In fact Mp_n & O_n form a __dual pair__ acting on $L^2\mathbb{R}^n \circlearrowleft O_n$

- $n=1$ $O_1 = \mathbb{Z}/2$, Mp_2 preserves subspaces of even & odd functions & each one irreducible ....]

cf R. Howe, On the role of the Heisenberg group in harmonic analysis Bull AMS 3 # 2 1980

"Odd" analogue

Replace skew by symmetric!

$V, \langle , \rangle$ vector space with nondegenerate symmetric form [assume V complex]

$\rightsquigarrow$ replace relation $[x,y] = \langle x,y \rangle$ or $xy - yx = \langle x,y \rangle$ we had in symplectic case by

$$xy + yx = 2 \langle x,y \rangle$$

$$\iff x^2 = \langle x,x \rangle.$$

ie generate an algebra from V with this relation: Clifford algebra

$$Cl\, V = \underset{\text{tensor algebra}}{TV} / \langle xy + yx - 2\langle x,y \rangle \rangle.$$

Just like taking an enveloping algebra: mod out tensor algebra by inhomogeneous quadratic relation....

PBW theorem: $Cl\, V$ is isomorphic as a vector space to corresponding homogeneous relation $TV / \langle xy + yx \rangle = \Lambda^\bullet V$

.... have a basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{\substack{k=1,\ldots,n \\ i_1 < i_2 < \ldots < i_k}}$

– in particular $\dim = 2^{\dim V}$.

In $Cl\,V$ case ideal is homogeneous in $\mathbb{Z}/2$ graded sense (deg 2 − deg 0)
$\Rightarrow$ get a $\mathbb{Z}/2$ grading $Cl\,V = Cl_+ \oplus Cl_-$
Cl_+ is a subalgebra of dim 2^{n-1}.

We'll construct modules for Cl just as for the Heisenberg algebra:
Suppose $W \subset V$ is an isotropic subspace, ie $\langle\,\rangle|_W = 0$. $\Longrightarrow \Lambda^\bullet W \subset Cl\,V$ subalgebra. $\Rightarrow$ can construct a rep

$$\mathrm{Ind}_{\Lambda^\bullet W}^{Cl\,V} \mathbb{C} = Cl\,V \underset{\Lambda^\bullet W}{\otimes} \mathbb{C}$$

$$= Cl\,V \cdot v_0 \quad \text{where} \quad w \cdot v_0 = 0 \;\; w \in W,$$
no other relations

.... skew analog was pick any $W \subset V$ isotropic & consider $\mathrm{Sym}\,W \subset U\mathfrak{h}_V$,
$\rightsquigarrow$ induced representation $U\mathfrak{h}_V \underset{\mathrm{Sym}\,W}{\otimes} \mathbb{C}$

Why isotropic? otherwise can't have W (or $\mathrm{Sym}\,W$ / $\Lambda^\bullet W$) act by 0 since

$$uv \pm vu = \langle u,v\rangle \cdot 1 \quad \text{is nonzero.}$$

If W not <u>maximal</u> isotropic $\Rightarrow \exists\, u \in V$
$\langle u,w\rangle = 0 \quad \forall w \in W$

$\Rightarrow$ have a $\neq 0$ homomorphism of reps from $\mathrm{Ind}_W^V \mathbb{C}$ to $\mathrm{Ind}_{W\oplus\mathbb{C}u}^V \mathbb{C}$

so former is not irreducible!

So might as well....

Two cases: 1. dim V even $\Rightarrow$ write $V \simeq W \oplus W'$, W' also isotropic

2. dim V odd $\Rightarrow$ $V \simeq \underbrace{W \oplus W'}_{\text{isotropic}} \oplus \underbrace{\mathbb{C}}_{\neq 0 \text{ inner prod}}$

Case 1: rep $\mathcal{H}_W \simeq \Lambda^\bullet W' \circlearrowleft Cl\, V$

Prop $Cl(V) \simeq \mathrm{End}\, \mathcal{H}_W$ matrix algebra

- just check $\Lambda^\bullet(W \oplus W^*) \simeq \mathrm{End}\, \Lambda^\bullet W$ as vector spaces

In particular $\mathcal{H}_W$ unique irrep up to isomorphism! "Stone-von Neumann"

... can actually realize $\mathcal{H}_W$ as a left ideal inside $Cl\, V$...

Cl_+ preserves decompostn $\underset{\mathcal{H}_W^+}{\Lambda^{\text{even}} W'} \oplus \underset{\mathcal{H}_W^-}{\Lambda^{\text{odd}} W'}$

$\rightsquigarrow$ find $Cl_+ \simeq \mathrm{End}\, \mathcal{H}_W^+ \oplus \mathrm{End}\, \mathcal{H}_W^-$

$$\left(\begin{array}{c|c} ++ & -+ \\ \hline +- & -- \end{array}\right)$$

[**Case 2** $Cl(V) \simeq \operatorname{End} \mathcal{H}_W \oplus \operatorname{End} \mathcal{H}_{W'}$.
while $Cl_+ \simeq \operatorname{End} \mathcal{H}_W$]

Like any associative algebra, we can consider it as a Lie algebra

Construction $\mathfrak{so}_n$ is a Lie subalgebra of Cl_+

First realize $\mathfrak{so}(V) \simeq \Lambda^2 V$
via $a \wedge b \longmapsto \varphi_{a\wedge b} \in \operatorname{End} V$
$\varphi_{a\wedge b}(v) = 2(\langle b,v\rangle a - \langle a,v\rangle b)$
(just saying skew-symmetric matrices are skew-symmetric).

Now map $\Lambda^2 V \longrightarrow Cl_+ V$
via skew symmetrization:
$a \wedge b \longmapsto \frac{1}{2}(a\cdot b - b \cdot a)$
$= a \cdot b - \langle a,b\rangle 1$, check
it's a Lie algebra embedding.

$\Rightarrow$ n even find two reps S_+, S_- of dim 2^{n-1}
n odd single rep S
These are all irreducible!

Now can ask if they come from rep of SO_n.

In fact build a group out of Clifford algebra:

let $(\,)^* : Cl \circlearrowleft$ be

$$(v_1 \cdots v_r)^* = (-1)^r v_r \cdots v_1$$

ie generated by $vw \longmapsto -wv$.

$$\begin{tikzcd} \pm 1 \arrow[r, bend left] & \mathrm{Spin}\, V = \left\{ x \in Cl_+ : \begin{array}{l} xx^* = 1 \ \& \\ xVx^* \subset V \end{array} \right\} \arrow[d] \\ & SO(V) \ni \{ v \mapsto xvx^{-1} = xvx^* \} \end{tikzcd}$$

double cover which acts on spin reps.

eg $V = \mathbb{R}^3$ $\qquad Cl\, V \simeq \begin{array}{c} \mathbb{R} \\ \left.\begin{array}{c} \mathbb{R}^3 \\ \mathbb{R}^3 \\ \mathbb{R} \end{array}\right\} Cl_+ \end{array}$

$Cl_+ \simeq \mathbb{H}$ quaternions!

$so_3 \simeq \Lambda^2 V \simeq \mathbb{R}^3 \longleftrightarrow \mathbb{H}$

Group of unit quaternions $= SU_2 \twoheadrightarrow SO_3$

Spin representation is $\mathbb{C}^2$ ...

(real version of above complex Clifford story..)