HOMEWORK 1

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There will be weekly homework assignments due each Wednesday at the beginning of class. Please work the problems neatly and staple your pages together. There is no need to copy over the problem or hand in the problem sheet. Please number the problems. *Include your name at the top of your homework!* Do not show scratch work. Try the problems on your own first. Then feel free to discuss them and work together with classmates, friends, parents, etc. However, I expect you to write up your own solutions to the problems. Please come and discuss the problems (and the class generally) with me during office hours. Feel free to make an appointment with me at other times.

Do the best you can on these problems. Please write your scratch work on scratch paper and only hand in coherent, readable arguments and calculations. You will have to produce good mathematical writing on your tests, prelims, and in your future mathematical writing, so why not practice now on these homeworks? Good mathematical writing is concise, so don't write volumes of material. Some of the problems are computational, others conceptual. Some may involve ideas you are not familiar with. (Here is one place where your classmates may be able to help you.) Often I leave problems open-ended. Feel free to explore, but please don't write a book. Some are meant to be challenging, so do not get discouraged if you find them difficult. I certainly don't expect anyone to do all of the problems or even to come close. They are a guide for your learning of the material in the lectures and the book.

You should be reading Warner and Guillemin/Pollack along with the class, even if I do not give specific reading assignments. Please skim through the book immediately to see if you are comfortable with it and think it is at the right level for you. (We will only cover Chapters 1, 2, and 4 of Warner.) And don't forget Milnor—it's a wonderful book and a wonderful piece of mathematical writing. If you think you are going to have trouble with the class, it is best to find this out early. Feel free to ask questions about the book during class or office hours. You should now be reading the first

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chapter in each of the text books. There are plenty of problems in each, and I suggest you try some of them.

All vector spaces (and eventually manifolds) studied in this class are assumed to be finite dimensional. The coefficients are real.

1. Problems

Problem 1.1. Consider the n-sphere $S^n \subset \mathbb{A}^{n+1}$ defined by

$$S^n := \{(x^1, \dots, x^{n+1}) \in \mathbb{A}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$$

Cover S^n with two coordinate charts (called *stereographic projections*) as follows. Let

$$\begin{aligned} n &= (0, \dots, 0, +1) \\ s &= (0, \dots, 0, -1) \end{aligned}$$

be the north and south pole. Identify $\mathbb{A}^n$ with its image under

$$\begin{aligned} \mathbb{A}^n &\longrightarrow \mathbb{A}^{n+1} \\ (x^1, \dots, x^n) &\longmapsto (x^1, \dots, x^n, 0) \end{aligned}$$

For $p \in S^n$ let $x(p)$ be the intersection of the line through p and n with $\mathbb{A}^n$, and let $y(p)$ be the intersection of the line through p and s with $\mathbb{A}^n$. Where are x and y coordinate maps? What is the transition function between the two coordinate systems?

Problem 1.2. Suppose X, Y are manifolds and $f : X \to Y$ a smooth map.

(a) Prove that $X \times Y$ is also a manifold.

(b) Show that the *graph* $\Gamma(f) \subset X \times Y$ of f, defined by

$$\Gamma(f) := \{(x, y) \in X \times Y : y = f(x)\},$$

is a manifold.

(c) Now suppose $U \subset X$ is an open subset. Define a manifold structure on U.

Problem 1.3. There are three different ways in which functions correspond to shapes. Let X, Y be sets and $f : X \to Y$. (You might prefer to think that X, Y are topological spaces and f is continuous, but it doesn't matter for this part.) The *graph* of f, which is a subset of $X \times Y$, is defined in the previous problem. The *image* of f is $f(X) \subset Y$. The fiber of f at $c \in Y$ is $f^{-1}(x) \subset X$. Graphs were discussed in the previous problem; now we consider fibers and images, all in the context of smooth maps of smooth manifolds.

(a) Suppose $f : \mathbb{A}^3 \to \mathbb{R}$ is a smooth function. Define $M_c = f^{-1}(c)$ for all $c \in \mathbb{R}$. Is M_c necessarily a manifold? Think carefully about what that statement means. For a fixed f what can you say about the set of c for which M_c is a manifold? Try many examples. You might also want to try this problem with $\mathbb{A}^2$ replacing $\mathbb{A}^3$.

(b) Repeat with $f : \mathbb{R} \to \mathbb{A}^3$ and $M = f(\mathbb{R}) \subset \mathbb{A}^3$. Here there is no parameter ('c' in the previous), so you'll have to vary the map f.

Problem 1.4. (a) Suppose A is an n dimensional affine space with associated vector space V. Let $\gamma : (a,b) \to A$ be a smooth curve, where $(a,b) \in \mathbb{R}$ is an open set. For $t \in (a,b)$ define the tangent vector $\dot\gamma(t) \in V$.

(b) Take $n = 2$ and $A = \mathbb{A}^2$ so we write $\gamma(t) = \big(x(t), y(t)\big)$. What is the formula for the tangent vector in terms of the real-valued functions x, y? (What is the vector space V?)

(c) Suppose the image of γ lies in the open set $\mathbb{A}^2 \setminus \{(x,0) : x \geq 0\}$. Introduce polar coordinates (r, θ) on this open set and write the curve as $\big(r(t), \theta(t)\big)$. How precisely are the functions r, θ defined? What is the tangent vector to the curve in terms of the functions r, θ?

Problem 1.5. Let X denote the set of affine lines in $\mathbb{A}^2$. Topologize X and show that it is a topological manifold. What is $\dim X$? Is X connected? Is X compact? Is X simply connected? Can you recognize X as a familiar topological manifold: do you know a familiar topological manifold which is homeomorphic to X?

Problem 1.6. Recall that $GL_n\mathbb{R}$ is the open subset of $n \times n$ real matrices which are invertible, so it inherits a manifold structure from that of the vector space of $n \times n$ matrices. Show that multiplication and inversion are smooth maps

$$GL_n\mathbb{R} \times GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$$
$$GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$$

Repeat for the group $GL_n\mathbb{C}$ of invertible complex matrices. This proves that $GL_n\mathbb{R}$ and $GL_n\mathbb{C}$ are *Lie groups*. At some point during the semester I strongly recommend reading Chapter 3 of Warner for some basics on Lie groups.