# HOMEWORK 2

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I deliberately pose some of the problems in an open-ended fashion. Feel free to explore. Try some examples, make your own hypotheses, try to compute further examples or make theoretical arguments to verify your hypotheses. In other words, use these problems to practice the skills you use to do research. You are welcome to look in books, though I encourage you to emphasize exploration first as well as after.

**Problem 0.1.** Review the general form of the chain rule; here you will use it in a specific example. Define $f, g : \mathbb{A}^3 \to \mathbb{A}^3$ by $f(x, y, z) = (x^2, y^2, z^2)$ and $g(x, y, z) = (yz, xz, xy)$.

(a) Write a formula for $g \circ f$ and compute $d(g \circ f)$ at a general point $(x, y, z)$ from the formula.

(b) Now compute $dg$ and $df$ separately and use the chain rule to compute $d(g \circ f)$. Compare your answer to that in part (a).

**Problem 0.2.** Prove the following assertions, which should look familiar.

(a) Suppose $U \subset \mathbb{A}^n$ is a connected open set and $f : U \to \mathbb{A}^m$ is a smooth function whose differential $df_x$ vanishes for all $x \in U$. Prove that $f$ is constant.

(b) Let $U, V \subset \mathbb{A}^n$ be open subsets and $f : U \to \mathbb{A}^n$ and $g : V \to \mathbb{A}^n$ be smooth maps such that the compositions $f \circ g$ and $g \circ f$ are defined and equal to the identity map. Prove that for each $x \in U$ the differential $df_x$ is an invertible map.

(c) Let $U$ be a connected open subset of an affine space $A$ and $f : U \to B$ a smooth map to an affine space $B$. Prove that $f$ extends to an affine map $A \to B$ if and only if the differential $df : U \to \mathrm{Hom}(V, W)$ is constant. Here $V, W$ are the vector spaces associated to the affine spaces $A, B$ and $\mathrm{Hom}(V, W)$ is the vector space of linear maps from $V$ to $W$.

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**Problem 0.3.** (a) Define complex projective space $\mathbb{C}P^n$ as the manifold of equivalence classes

$$\mathbb{C}P^n = \{[z^0, z^1, \dots, z^n] : (z^0, z^1, \dots, z^n) \neq (0, 0, \dots, 0)\} /\!\sim,$$

where

$$[z^0, \dots, z^n] \sim [z'^0, \dots, z'^n] \qquad \text{if and only if} \qquad z'^i = \lambda z^i$$

for some nonzero complex number $\lambda$. Show that $\mathbb{C}P^n$ is a manifold. (Consider $U_i = \{[z^0, \dots, z^n] : z^i \neq 0\}$.)

(b) Construct a diffeomorphism between $\mathbb{C}P^1$ and the standard 2-sphere.

(c) Identify the 3-sphere with the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}.$$

(Think of $\mathbb{C}^2$ as an affine space for this.) Identify the 2-sphere as $\mathbb{C}P^1$. What is the inverse image of a point? What can you say about the inverse image of two distinct points? The map $f$ is called the *Hopf fibration*.

**Problem 0.4.** In this problem you will fill in details of the construction of projective space that was sketched in lecture; it is a more abstract approach than the previous problem. Let $V$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and denote by $\mathbb{P}V$ the set of lines in $V$. Recall that a *line* is a 1-dimensional vector space, so a line in $V$ is a one-dimensional subspace of $V$.

(a) Let $L \subset V$ be a line and $W \subset V$ a complementary subspace, i.e., $V = L \oplus W$. Define

$$\begin{aligned} \phi_{L,W}\ \mathrm{Hom}(L; W) &\longrightarrow \mathbb{P}V \\ T &\longmapsto L_T \end{aligned}$$

where $L_T = \{\ell + T\ell : \ell \in L\}$ is the graph of $T$. We can identify $L_T$ as the image of the linear map $\mathrm{id}_L + T\, L \to L \oplus W = V$. Show that the image of $\phi_{L,W}$ is $\mathbb{P}V \setminus \mathbb{P}W$.

(b) Now consider a second pair $(L', W')$ and the corresponding $\phi_{L',W'}$. We now have two parametrizations of $\mathbb{P}V \setminus (\mathbb{P}W \cup \mathbb{P}W')$, so can compare by an overlap isomorphism

$$f\, U \longrightarrow U',$$

where $U \subset \mathrm{Hom}(L; W)$ and $U' \subset \mathrm{Hom}(L'; W')$ are the images of $\mathbb{P}V \setminus (\mathbb{P}W \cup \mathbb{P}W')$ under the two parametrizations. Write a formula for the map $f$. Show that $f$ is smooth. (Hint: The formula involves $\pi^{L'} V \to L'$, the projection onto $L'$ with kernel $W'$.)

(c) Use the parametrizations $\phi_{L,W}$ to topologize $\mathbb{P}V$ and construct an atlas, so make $\mathbb{P}V$ a smooth manifold. What is its dimension? Prove that a linear map $V' \to V$ induces a smooth map $\mathbb{P}V' \to \mathbb{P}V$. What, then, is a projective line in $\mathbb{P}V$? What is the collection of all such?