# HOMEWORK 3

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**Problem 0.1.** Let $x^1, \ldots x^n$ and $y^1, \ldots, y^n$ be two local coordinate systems (charts) on a smooth manifold, and suppose the domains agree. Let $f$ be a smooth real-valued function defined on this common domain.
Show

$$\frac{\partial f}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial f}{\partial x^i}.$$

Verify from (a) and other equations that

$$df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial y^j} dy^j.$$

Compute

$$\frac{\partial^2 f}{\partial y^j \partial y^k}$$

in terms of partial derivatives of $f$ in the $x$-coordinate system. Does

$$\frac{\partial^2 f}{\partial y^j \partial y^k} dy^j dy^k$$

behave nicely under coordinate change? You'll have to invent a multiplication rule for the differentials to answer this, so try to find one so that this expression does behave nicely.

**Problem 0.2.** Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1\}.$$

There is an obvious inclusion $i\, S^2 \to \mathbb{A}^3$. Show that the differential $di_p$ at any point $p \in S^2$ is an injection $di_p\, T_pS^2 \to \mathbb{R}^3$ and identify the image.

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On the upper hemisphere $\{z > 0\}$ we can consider the functions $x, y$ to be coordinate function. As a second coordinate system we take spherical coordinates $\theta, \phi$ defined by solving the equations

$$x = \sin\phi\cos\theta$$
$$y = \sin\phi\sin\theta.$$

Identify a (maximal) subset of the upper hemisphere on which $\theta, \phi$ is a coordinate system. (You may want to translate: replace $\theta, \phi$ by $\theta - \theta_0$, $\phi - \phi_0$ for some $\theta_0, \phi_0$.) On that subset express the vector field $\partial/\partial x$ in terms of $\partial/\partial\theta$ and $\partial/\partial\phi$. Also, compute $di(\partial/\partial x)$ as a vector field on a subset of $\mathbb{A}^3$.

**Problem 0.3.** This problem is a standard and important corollary of the inverse function theorem [which you'll need to read about in the texts - we will discus in Monday's class]. It states a condition under which we can solve an equation of two variables implicitly for one variable as a function of the other.

Suppose $X, Y, Z$ are manifolds and $F\ X \times Y \to Z$ a smooth map with $F(x_0, y_0) = z_0$ for some $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Assume that the restriction of the differential $dF_{(x_0,y_0)}$ to $T_{y_0}Y \subset T_{(x_0,y_0)}(X \times Y)$ is an isomorphism onto $T_{z_0}Z$. Prove that there exists a neighborhood $U$ of $x_0$ and $V$ of $y_0$ and a smooth function $f\ U \to V$ such that

$$F\bigl(x, f(x)\bigr) = z_0$$

for all $x \in U$. (More generally, we can find a function $f_z U \to V$ which solves the equation $F\bigl(x, f_z(x)\bigr) = z$ for $z$ in a neighborhood of $z_0$.)

**Problem 0.4.** Fix positive numbers $r$ and $R$ with $r < R$. Let the torus $T$ be the surface of revolution in $\mathbb{A}^3$ (with coordinates $x, y, z$) obtained by revolving the circle

$$y = 0, \quad (x - R)^2 + z^2 = r^2$$

about the $z$-axis.

Show that $T$ is a 2-manifold.

Define the *Gauss map* $g\ T \to S^2$ to the unit sphere in $\mathbb{A}^3$ by mapping a point $p \in T$ to the unit normal vector to $T$ at $p$, viewed as a point of $S^2$. (Here I am relying on your geometric intuition, not on definitions we have discussed in this class.) Show that $g$ is smooth. Compute its differential in some coordinate system.

**Problem 0.5.** (a) Continuation of Problem 3 part c from Homework 2: Prove that the map

$$f\ S^3 \longrightarrow S^2$$
$$(z^1, z^2) \longmapsto [z^1, z^2]$$

is a submersion, i.e., its differential has maximal rank at each point.

(b) Continuation of Problem 4 from Homework 2: Construct a natural isomorphism (take this to mean that it doesn't depend on choices)

$$T_L(\mathbb{P}V) \to \mathrm{Hom}(L, V/L).$$