HOMEWORK 7

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Problem 0.1. (a) Let V be a finite dimensional real vector space. Recall that an *inner product* on V is a function $\langle -, - \rangle \; V \times V \to \mathbb{R}$ which is linear in each variable separately, symmetric, and positive definite: for $\xi, \xi_1, \xi_2, \eta \in V$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \langle \xi_1 + \lambda\xi_2, \eta \rangle &= \langle \xi_1, \eta \rangle + \lambda \langle \xi_2, \eta \rangle \\ \langle \xi, \eta \rangle &= \langle \eta, \xi \rangle \\ \langle \xi, \xi \rangle &> 0 \qquad \text{if } \xi \neq 0 \end{aligned}$$

(b) Show that the space of maps $V \times V \to \mathbb{R}$ which satisfy the first two equations above is a vector space. What is its dimension (in terms of $\dim V$)? Show that the subset of maps which in addition satisfy the positive definiteness condition is *convex*, i.e., the line segmet joining any two points in the subset is contained in the subset.

(c) A *Riemannian metric* on a smooth manifold X is a smoothly varying assignment of inner products on the tangent spaces T_pX. How do we formalize 'smoothly varying' in the previous sentence?

(d) Construct a Riemannian metric on $U \subset X$ if U is the domain of a coordinate chart $(U; x^1, \ldots, x^n)$. Use a partition of unity to construct a Riemannian metric on X.

Problem 0.2. Suppose that X is a Riemannian manifold, that is, a smooth manifold endowed with a Riemannian metric. (We know they exist by the previous problem.) Let $\gamma \; [0, 1] \to X$ be a smooth path. Define the length of γ to be

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt,$$

where the norm $|\dot{\gamma}|$ is defined as $\langle \dot{\gamma}, \dot{\gamma} \rangle^{1/2}$. Given $x_0, x_1 \in X$ define the distance

$$d(x_0, x_1) = \inf_{\gamma} L(\gamma),$$

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where γ ranges over all parametrized paths $\gamma[0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. Prove that d is a metric in the sense of metric spaces. This proves that the topological space underlying a smooth manifold is metrizable (so the topology is not too pathological).

Problem 0.3. For a Lie group G with universal covering Lie group $\widetilde{G}$, show that $\pi_1(G)$ is a subgroup of the *center* of $\widetilde{G}$. (Hint: show the point wise product in either order can be deformed to a composition of paths).

Problem 0.4.
(a) Let M be a smooth manifold, $\pi E \to M$ a real vector bundle, and $E' \hookrightarrow E$ a subbundle. This means that $\pi|_{E'} E' \to M$ is a vector bundle and $E'_p \subset E_p$ is a linear subspace for all $p \in M$. Prove that the quotient bundle $E'' = E/E'$, defined by $E''_p = E_p/E'_p$, is a vector bundle over M.
(b) Suppose

$$0 \longrightarrow E \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0$$

is a short exact sequence of vector spaces. Recall this means i is injective, j is surjective, and $\ker j = i(E')$. A *splitting* is a linear map $s E'' \to E$ which is right inverse to j. Prove that the space of splittings is an affine space over $\mathrm{Hom}(E''; E')$. Prove also that a splitting is equivalent to a left inverse to i.
(c) Let A be a real affine space and $a_1, \dots, a_n \in A$. Suppose $\rho_1, \dots, \rho_n \in \mathbb{R}$ satisfy $\rho_1 + \cdots + \rho_n = 1$. Make sense of $\rho_1 a_1 + \cdots + \rho_n a_n \in A$.
(d) Now suppose M is a smooth manifold and

$$0 \longrightarrow E \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0$$

a short exact sequence of vector bundles over M. (This means that over each $p \in M$ we have a short exact sequence.) Use a partition of unity argument to construct a splitting of this sequence.

Problem 0.5. Let G be a connected Lie group and U any neighborhood of the identity. Show that any element of G can be written as a finite product of elements of U. In particular, U generates the group G. (Hint: show the complement of the locus generated by U is closed, and argue by contradiction using translates of U).

Problem 0.6. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. For a continuous map $F : M \to N$ denote by $F^* : C(N) \to C(M)$ the pullback of functions.

(a) Show that F^* is a linear map (in fact preserving multiplication, i.e., a map of algebras).
(b) If M, N are smooth manifolds show that $F^*(C^\infty(N)) \subset C^\infty(M)$ if and only if F is smooth.
(c) If $F : M \to N$ is a homeomorphism, show it is a diffeomorphism if and only if F^* restricts to an isomorphism of $C^\infty(N)$ with $C^\infty(M)$.