INTRO TO HOMOLOGICAL ALGEBRA AND SPECTRAL SEQUENCES MINICOURSE

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Abstract. Focus on Ext, Tor, and the Serre and Universal Coefficient Spectral Sequences. These are very useful for calculating homology/cohomology of topological spaces. There will also be some group cohomology calculations. Adrian will lecture on how to build a spectral sequence from a filtration.

1. Monday Exercises

Exercise 1.1. Show that there exists a chain homotopy equivalence $f : X \to X'$ where $f_n : X_n \to X'_n$ is not an isomorphism by considering the chain complexes

$$\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \cdots$$
$$\cdots \to 0 \to 0 \to 0 \to 0 \cdots$$

Exercise 1.2. Show that quasi-isomorphism is not symmetric by considering the chain complexes of abelian groups

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \cdots$$
$$\cdots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \cdots$$

This also gives an example of a quasi-isomorphism that is not a chain homotopy equivalence.

Exercise 1.3. Let $R = k[x, y]$. Show that the chain complexes of R-modules

$$0 \to R \oplus R \xrightarrow{h} R \to 0$$
$$0 \to R \xrightarrow{0} k \to 0$$

have the same homology but are not quasi-isomorphic. The map h is defined by

$$h(f(x, y), g(x, y)) = xf(x, y) + yg(x, y)$$

Exercise 1.4. Show that a short exact sequence $0 \to A \to B \to C$ of chain complexes of R-modules givess rise to a long exact sequence of homology groups

$$\cdots \to H_k(A) \to H_k(B) \to H_k(C) \to H_{k-1}(A) \to H_{k-1}(B) \to H_{k-1}(C) \to H_{k-2}(A) \to \cdots$$

Exercise 1.5. Show that the category of chain complexes of R-modules up to quasi-isomorphism (that is, you have the same objects of $\mathsf{Ch}(R)$, but you invert all the quasi-isomorphisms) is not abelian, but is instead triangulated. This category is called the derived category of R.

Exercise 1.6. If A is a bounded chain complex of finitely generated abelian groups (that is, all but finitely many A_n are 0, and all the A_n are finitely generated), then we can define the Euler characteristic

$$\chi(A) = \sum (-1)^n \text{rank}(A_n)$$

Show that Euler characteristic can also be computed on the level of homology, that is

$$\chi(A) = \sum (-1)^k \text{rank}(H_k(A))$$

Exercise 1.7. Watch the 1980 film, "It's My Turn", an American romantic comedy-drama film starring Jill Clayburgh, Michael Douglas and Charles Grodin.

Exercise 1.8. Prove the splitting lemma: For any short exact sequence of R-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, then the following are equivalent:

(1) There exists a map $u : B \to A$ such that $u \circ f = id_A$.
(2) There exists a map $v : C \to B$ such that $v \circ g = id_B$.
(3) $B \simeq A \oplus C$.

Exercise 1.9. Show that the splitting lemma does not hold in the category of groups. (**Hint:** Consider the SES of groups $0 \to A_3 \xrightarrow{i} S_3 \xrightarrow{det} \mathbb{Z}/2 \to 0$)

2. Tuesday Exercises

Exercise 2.1. Prove the following properties of Tor and Ext.

(1) $\mathrm{Tor}_1^R(\oplus_i M_i, N) \cong \oplus_i \mathrm{Tor}_1^R(M_i, N)$
(2) $\mathrm{Tor}_1^R(M, N) = 0$ if M (or N) is free.
(3) $\mathrm{Tor}(\mathbb{Z}/n, A) = ker(A \xrightarrow{\cdot n} A)$ for A an abelian group.
(4) For each short exact sequence $0 \to A \to B \to C \to 0$, we have a natural exact sequence

$$0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$

(5) $\mathrm{Tor}_1^R(M, N) \cong \mathrm{Tor}_1^R(N, M)$.
(1) $\mathrm{Ext}_R^1(\oplus_i M_i, N) \cong \oplus_i \mathrm{Ext}_R^1(M_i, N)$
(2) $\mathrm{Ext}_R^1(M, N) = 0$ if M is free (or in general, projective).
(3) $\mathrm{Ext}(\mathbb{Z}/n, A) = A/nA$ for A an abelian group.
(4) For each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence

$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to \mathrm{Ext}_R^1(C, N) \to \mathrm{Ext}_R^1(B, N) \to \mathrm{Ext}_R^1(A, N) \to 0$$

(5) Similarly, for each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \mathrm{Ext}_R^1(M, A) \to \mathrm{Ext}_R^1(M, B) \to \mathrm{Ext}_R^1(M, C) \to 0$$

Exercise 2.2. Suppose that X is a topological space such that $H_k(X)$ is free and finitely generated for all k. Prove that $H^k(X) \cong H_k(X)$.

(a) Compute $H^k(\mathbb{C}P^\infty)$ for all n.
(b) Compute $H^k(S^n)$ for all k, n.

Exercise 2.3. Suppose that R is a field. Then show that $H_*(X) \otimes_R M \cong H_*(X; M)$, and $H^*(X; M) \cong \mathrm{Hom}(H_*(X), M)$.

Exercise 2.4. Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 2.5. Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/m)$ for all k and for any odd integer $m \geq 3$.

Exercise 2.6. Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z})$ for all k.

Exercise 2.7. Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 2.8. Compute $H^k(\mathbb{R}P^\infty; \mathbb{Z})$ for all k.

Exercise 2.9. Compute $H^k(\mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 2.10. Let G be an abelian group. Let $M(G, n)$ be a Moore space, that is to say it satisfies

$$\widetilde{H_k}(M(G,n)) = \begin{cases} G & k = n \\ 0 & else \end{cases}$$

(a) Construct a CW complex $M(\mathbb{Z}/m, n)$ having 3 cells.
(b) Consider the map $f : M(\mathbb{Z}/m, n) \to S^{n+1}$ that collapses the n-skeleton. Use this map to show that the splitting in the Universal Coefficient Theorem is not natural.
(c) Consider the map $f \times id : M(\mathbb{Z}/m, n) \times M(\mathbb{Z}/m, n) \to S^{n+1} \times M(\mathbb{Z}/m, n)$. Use this map to show that the splitting in the Künneth Theorem is not natural.

3. Wednesday Exercises

Exercise 3.1. Where does the term Ext come from? Show that $\mathrm{Ext}^1_R(M, N)$ is isomorphic to the group of extensions up to equivalence of M by N, with addition given by the Baer sum:

Definition 3.2. If M and N are R-modules, then we say an extension of M by N is a SES of R-modules:

$$0 \to N \to E \to M \to 0$$

Definition 3.3. Two extensions $0 \to N \to E \to M \to 0$ and $0 \to N \to E' \to M \to 0$ are equivalent if there is a commutative diagram, middle is isomorphism

Proposition 3.4. The set of equivalence classes of extensions has a group structure coming from Baer sum. Given two extensions $0 \to N \to E \to M \to 0$ and $0 \to N \to E' \to M \to 0$, form the SES $0 \to N \oplus N \to E \oplus E' \to M \oplus M \to 0$. Pull back along the diagonal map $M \to M \oplus M$, and then push out along the sum map $A \oplus A \to A$ to obtain the Baer sum $0 \to N \to E'' \to M \to 0$

Exercise 3.5. Generalize the previous exercise to show that $\mathrm{Ext}^n_R(M, N)$ is isomorphic to the group of extensions of M by N of length n up to equivalence.

Definition 3.6. If M and N are R-modules, then we say an extension of M by N of length n is an exact sequence of R-modules:

$$0 \to N \to E_{n-1} \to \cdots E_0 \to M \to 0$$

Exercise 3.7. Show that we have an associative and unital map called the Yoneda product:

$$\mathrm{Ext}^n_R(N, P) \otimes \mathrm{Ext}^m_R(M, N) \to \mathrm{Ext}^{m+n}_R(M, P)$$

by considering the isomorphism bewteen $\mathrm{Ext}^n_R(M, N)$ and the group of extensions of M by N of length n up to equivalence.

Exercise 3.8. Let M be a $\mathbb{Z}G$-module. Show that $H^0(G; M) = M^G$, the G-fixed points of M. Also show that $H_0(G; M) = M_G$, the coinvariants of M. In other words, M_G is the quotient of M by the submodule generated by elements of the form $g \cdot m - m$.

Exercise 3.9. Show that if k is a field of characteristic p, and G is $\mathbb{Z}/p$, then

$$H^*((G)^n, k) = \begin{cases} \mathbb{F}_p[x_1, \cdots, x_n] & |x_i| = 1,\ p = 2 \\ \Lambda[x_1, \ldots x_n] \otimes \mathbb{F}_p[y_1, \cdots, y_n] & |x_i| = 1, |y_i| = 2,\ p \neq 2 \end{cases}$$

4. Friday Exercises

Exercise 4.1. Show that the cohomology of X with local coefficients in $\mathbb{Z}[\pi(X)]$ is isomorphic to the cohomology of the universal cover of X, $\tilde{X}$. That is,

$$H_n(X; \mathbb{Z}[\pi(X)]) \cong H_n(\tilde{X})$$

Exercise 4.2. Show that $H^n(G; k) \cong H^n(BG; k)$.

Exercise 4.3. Let $1 \to N \to G \to G/N \to 1$ be a SES of groups, and let M be a G-module (M is an abelian group with a G-action that distributes over addition).

Show that we have the Lyndon-Hochschild-Serre spectral sequence, with

$$E^2_{p,q} = H^q(G/N; H^q(N; M)) \Rightarrow H^{p+q}(G; M)$$

Exercise 4.4. Show that a (Serre) fibration $F \to E \to B$ induces a long exact sequence of homotopy groups

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_0(E)$$

Exercise 4.5. Prove(recover) the Hurewicz isomorphism using the path fibration.

Exercise 4.6. Prove(recover) the Eilenberg-Zilber Theorem.

Exercise 4.7. Play around with the fibration $S^n \to D^n \to S^{n+1}$.

Exercise 4.8. Play around with the Hopf fibration $S^1 \to S^3 \to S^2$.

Exercise 4.9. Compute the homology and cohomology of the degree p map from $S^n \to S^n$. (assuming we have replaced it with a Serre fibration).

Exercise 4.10.

Definition 4.11. Let $V_2(\mathbb{R}^{n+1})$ be the space of orthogonal pairs of vectors in $\mathbb{R}^{n+1}$.

(1) Show we have a Serre fibration $S^2 \to V_2(\mathbb{R}^{n+1}) \to S^n$
(2) Compute $H^*(V_2(\mathbb{R}^{n+1}))$.

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