Symmetric Spaces
Exercises Day 1

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Many of these exercises are pulled from statements in Eberlein. Assume M is a complete Riemannian manifold of nonpositive curvature.

1. Let p be a point in M. Let $v \in T_pM$ and $\xi \in T_v(T_pM)$ be arbitrary. Then:

   (a) $||d\exp_p(\xi) \geq ||\xi||$, where T_vT_pM has the standard Euclidean metric inherited from T_pM.

   (b) If $\sigma : [a,b] \to T_pM$ is any smooth curve then the length of σ is at most the length of $\exp_p \circ \sigma$. In particular if M is simply connected, $d(\exp_p(v), \exp_p(w)) \geq ||v - w||$ for any two vectors $v, w \in T_pM$.

   Hint: use Jacobi vector fields and the Jacobi equation.

2. Let p, q, r be distinct points of a simply connected M and let a, b, c be the lengths of the sides of a geodesic triangle with vertices p, q, r with opposite angles α, β, γ. Then:

   (a) (Law of Cosines) $c^2 \geq a^2 + b^2 - 2ab\cos(\gamma)$

   (b) (Double Law of Cosines) $c \leq b\cos(\alpha) + a\cos(\beta)$

   (c) If $\{p_n\}, \{q_n\}, \{r_n\}$ are sequences in M such that $d(p_n, q_n) \to +\infty$ but $d(q_n, r_n) \leq A$ as $n \to \infty$ for some positive constant A, then $\angle_{p_n}(q_n, r_n) \to 0$ as $n \to \infty$.

   (d) The angle sum of a geodesic triangle is at most π.

   Hint: use the previous exercise.

3. Show that the exponential map $\exp_p : T_pM \to M$ is a covering map. Conclude that if M is simply connected, the exponential map is a diffeomorphism and M is a unique geodesic space. (Hint: use the metric on M to define a metric on T_pM and apply the Hopf-Rinow theorem. Then show that any surjective local isometry of complete Riemannian manifolds is a covering map.)

4. Show the following Lie theoretic definitons for a symmetric space are equivalent. Let G be a real semisimple connected Lie group with Lie algebra $\mathfrak{g}$.

   (a) A *symmetric space for* G is a homoegenous space G/K where K is an open subgroup of the fixed point set of an involution $\sigma : G \to G$.

   (b) A *symmetric space decomposition of* $\mathfrak{g}$ is an involution $\theta : \mathfrak{g} \to \mathfrak{g}$.

   (c) A *symmetric space decomposition of* $\mathfrak{g}$ is a decompostion $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that
   $$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

5. Show that the isometry group of a geometric symmetric space is transitive.

6. Show that geometric symmetric spaces and Lie theoretic symmetric spaces coincide. You may assume that the isometry group of a geometric symmetric space is a Lie group.

7. Prove the lemma from the Cartan fixed point theorem.
   Lemma: Let $A \subset \tilde{M}$ be a compact set, and let $r : \tilde{M} \to \mathbb{R}$ be defined by $r(q) = \sup\{d(q, q')|q' \in A\}$. Then r assumes a minimum value at a unique point q_0 in $\tilde{M}$.