Symmetric Spaces
Exercises Day 2

Max Riestenberg

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Let N be a complete and connected Riemannian manifold. Then its isometry group $\mathrm{Isom}(N)$ has a Lie group structure compatible with the compact open topology. The following statement is proved in Helgason.

Lemma: If $\{\phi_n\}$ is a sequence of isometries such that $\{\phi_n(p)\}$ is bounded for some p in N, then there exists a subsequence $\{\phi_{n_k}\}$ and isometry ϕ such that ϕ_{n_k} converges to ϕ.

Assume M is a symmetric space of non-compact type. Let G be the identity component of its isometry group.

1. Let $p \in M$ and K be the stabilizer of p in G. Show that K is a maximal compact.

2. Show that the holonomy group of M at p is K.

3. Show directly that SL_n/SO_n is non-positively curved.

   (a) If p is the identity coset, show that $\mathfrak{k}$ is skew-symmetric matrices and $\mathfrak{p}$ is traceless symmetric matrices. What are σ_p and θ_p?

   (b) Show that the Killing form on $\mathfrak{p}$ is just the entry-wise dot product (up to a positive constant.)

   (c) Prove this formula for the curvature tensor:

   $$R(X, Y, Z) = -[[X, Y], Z]$$

   (d) Calculate (up to a positive constant) the sectional curvature

   $$K(u, v) = \langle R(u, v)v, u\rangle / \|u \wedge v\|$$

   on pairs of basis elements of $\mathfrak{p}$.

4. Let G be a Lie group with a bi-invariant metric. Show that G is a symmetric space with geodesic symmetry $S_p(x) = px^{-1}p$.

5. Show that every symmetric space $M = G/K$ of noncompact type imbeds into some $SL_n\mathbb{R}/SO_n\mathbb{R}$.

   (a) Show that the kernel of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the center of G.

   (b) Show that the center of G acts trivially on M. Hence if G is the identity component of the isometry group, its center is trivial.

   (c) Conlude that the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \subset SL_n\mathbb{R}$ is faithful. Show that it induces a well-defined diffeomorphism onto a complete totally geodesic submanifold of SL_n/SO_n. (Hint: pick a point and use the orbit map.) This map can fail to be an isometry, but only in a controlled way: explain this phenomenon.