Symmetric Spaces
Exercises Day 3

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Assume M is a complete simply connected manifold of nonpositive curvature.

1. Let G be a semisimple Lie group with finite center and no compact factors and K be a maximal compact subgroup. Show that the center of G acts trivially on G/K.

2. Show that the displacement function is convex.

3. An isometry ϕ of M is a *Clifford translation* if its displacement function is constant on M. Show that if ϕ is a nontrivial Clifford translation of M then $M = M_0 \times M_1$ has a Euclidean factor M_0 and ϕ is a Euclidean translation on M_0 and the identity on M_1.

4. Let ϕ be an elliptic isometry of M. Show that its fixed point set is a connected, complete, totally geodesic submanifold of M.

5. Let M/Γ be a closed manifold with $\Gamma \subset \mathrm{Isom}_0(M)$ discrete. Show that every nonidentity element in Γ is axial.

6. The canonical inner product ϕ_p on $\mathfrak{g}$ is defined by $\phi_p(X,Y) = -B(\theta_p X, Y)$ where B is the Killing form on $\mathfrak{g}$ and θ_p is the Cartan involution at $p \in M$. Show that if $X \in \mathfrak{k}$, $\operatorname{ad} X$ is a skew-symmetric transformation of $\mathfrak{g}$ relative to ϕ_p.

7. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $\mathfrak{g}$ decomposes into the ϕ_p-orthogonal direct sum of root spaces

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \text{ if } A \in \mathfrak{a} \text{ then } \operatorname{ad} A(X) = \alpha(A)X\}.$$

Prove all of the following properties of this root space decomposition:

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(b) If $\alpha \in \Lambda = \{\text{non-zero roots}\}$ then $-\alpha \in \Lambda$ and the Cartan involution θ_p is an isomorphism from $\mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$.

(c) The Cartan involution θ_p preserves $\mathfrak{g}_0$ so $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{k} + \mathfrak{a}$.

(d) If $A \in \mathfrak{a}$ then $\mathrm{Ad}(e^{tA}) = e^{t\alpha(A)} \cdot id_{\mathfrak{g}_\alpha}$ on $\mathfrak{g}_\alpha$.

(e) If $\alpha \neq -\beta$ then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

8. Find a symmetric space of noncompact type where $\mathfrak{a}$ is not all of $\mathfrak{g}_0$.

9. Show that $PSL_n(\mathbb{R})$ is the connected component of the isometry group of SL_n/SO_n.