Symmetric Spaces
Exercises Day 4

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Assume M is a symmetric space of noncompact type.

1. Let $\phi \in G = \mathrm{Isom}_0(M)$ be an axial or elliptic isometry of M. Let M_ϕ be the subset of M where ϕ achieves its minimum value. We know M_ϕ is complete and totally geodesic. Show that
   (a) $Z(\phi) = \{g \in G | g\phi = \phi g\}$ preserves and acts transitively on M_ϕ.
   (b) For every positive number R there exists a positive number a such that if $d(p, M_\phi) \geq a$ then $d_\phi(p) > R$.

2. Show that in the symmetric space associated to $SU(2,1)$, $\mathfrak{a}$ is not all of $\mathfrak{g}_0$.

3. Show that $PSL_n(\mathbb{R})$ is the connected component of the isometry group of SL_n/SO_n. Hint: try to show they have the same Lie algebra.

4. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let $\{H_\alpha, S_\alpha\}$ be the corresponding root system. Given $\alpha \in \Lambda$, let $X \in \mathfrak{g}_\alpha$ such that $\phi_p(X, X) = 2$ and write $X = K_\alpha + P_\alpha$ with $K_\alpha \in \mathfrak{k}$ and $P_\alpha \in \mathfrak{p}$. Show that:
   (a) $[A, K_\alpha] = \alpha(A)P_\alpha$
   (b) $[A, P_\alpha] = \alpha(A)K_\alpha$
   (c) $[K_\alpha, P_\alpha] = H_\alpha$
   (d) If $\phi_\alpha = e^{t_0 K_\alpha}$ with $t_0 = \pi/\sqrt{\langle H_\alpha, H_\alpha \rangle}$ then $\mathrm{Ad}(\phi_\alpha)$ realizes S_α: it has the same action on $\mathfrak{a}$.

5. Find explicit geodesics γ in SL_n/SO_n such that $F(\gamma)$ has a nontrivial summand $F_S(\gamma)$. More generally in any symmetric space of noncompact type M, given $p \in M$ and flat F containing p, how can you find a hyperbolic plane intersecting F in a geodesic through p? Hint: use roots.