Symmetric Spaces
Exercises Day 5

Max Riestenberg

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Assume M is a complete, simply-connected, nonpositively curved manifold.

1. Complete the proof that PSL_n is the identity component of the isometry group of SL_n/SO_n. Hint: isometries must take singular sets to singular sets. Relate the singular sets to subspaces of $\mathbb{R}^n$ and use the fundamental theorem of projective geometry.

2. Let G be a semisimple Lie group with trivial center and K a maximal compact. Show that G is the identity component of the isometry group of G/K. Hint: Since G acts transitively it reduces to showing that the subgroup K is the full stabilizer of the coset K in the isometry group. What does this look like at the level of Lie algebras?

3. We call two geodesics σ, γ *parallel* if $\gamma(\infty) = \sigma(\infty)$ and $\gamma(-\infty) = \sigma(-\infty)$. Show that any two parallel geodesics in M are contained in a common flat.

4. Let γ be any geodesic of M. Then for any point $p \in M$, show that there exists a unique geodesic σ of M such that $\sigma(0) = p$ and σ is asymptotic to γ.

5. Show that isometries induce homeomorphisms of $M(\infty)$.

6. It is possible to declare $M(\infty)$ to be isometric with some unit tangent sphere. Why is this an unnatural thing to do?

7. Now let M be a symmetric space of noncompact type and let p in M induce the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $X \in \mathfrak{p}$. Check that $Z(X) \cap \mathfrak{p}$ is a Lie triple system.

8. Show that for any $1 \leq k \leq \text{rank} = \dim(\mathfrak{a})$, there exists X in $\mathfrak{p}$ such that $\dim E_X = k$.

9. Let $\gamma : [0, \infty) \to X$ be a geodesic ray in any metric space, i.e. the map γ is an isometric embedding. Show that the *Busemann function* $f_\gamma : X \to \mathbb{R}$ defined by

$$f_\gamma(x) = \lim_{t \to \infty} d(x, \gamma(t)) - t$$

is well-defined.