Symmetric Spaces
Exercises Day 6

Max Riestenberg

June 4, 2018

Assume M is a symmetric space of noncompact type and $G = \mathrm{Isom}_0(M)$.

1. Let v be a regular unit vector at a point p in M. Let $g \in G$ be an isometry such that $g(p) = p$ and $dg(v) \in W(v)$. Then $dg(v) = v$. (In fact, dg fixes every vector tangent to the unique maximal flat containing v.)

2. Show that every parabolic subgroup acts transitively on M.

3. Show that given any pair of regular unit vectors v, w at points p, q respectively, there exists $g \in G$ such that $gp = q$ and $dgv \in W(w)$.

4. Let $p \in M$ with stabilizer the maximal compact K and induced Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let A_0 be any nonzero element of $\mathfrak{p}$ and $\phi \in K$ such that $\mathrm{Ad}(\phi)(A_0) = A_0$. Then for any $A \in E_{A_0}$, the intersection of all maximal abelian subspaces containing A_0, show that $\mathrm{Ad}(\phi)(A) = A$. (Hard. See p145 of Eberlein)

5. A point $x \in M(\infty)$ is *regular* if there exists a regular geodesic in its asymptotic class. Show that every geodesic in a regular ideal point x is regular.

6. A parabolic subgroup G_x is *minimal* if $P_y \subset P_x$ for some $y \in M(\infty)$ implies $P_y = P_x$, where $P_x = (G_x)_0$. Show that $x \in M(\infty)$ is regular if and only if G_x is minimal.

7. Let p be a point in M, F be a maximal flat in M containing p, and $z \in M(\infty)$. It makes sense to regard $F(\infty)$ as a subset of $M(\infty)$. Show that $G(z) \cap F(\infty)$ is finite, i.e. the G-orbit of z intersects $F(\infty)$ in finitely many points.