Symmetric Spaces
Exercises Day 7

Max Riestenberg

June 5, 2018

Assume M is a symmetric space of noncompact type and $G = \mathrm{Isom}_0(M)$.

1. If $W(x) = W(gx)$ for some $g \in G$ and $x \in M(\infty)$ then $gx = x$. Use this to show that
   (a) for every $x \in M(\infty)$, G_x is its own normalizer.
   (b) If $(G_x)_0 = (G_y)_0$ for points $x, y \in M(\infty)$ then $G_x = G_y$.
2. Show that $M(\infty)$ under the angle metric is complete.
3. Show that $\angle(x, y) = \min(\pi, Td(x, y))$ for all $x, y \in M(\infty)$. Use this to show that the cone topology and Tits topology coincide on $F(\infty)$ where F is a maximal flat in M.
4. Study the symmetric space $M = M_n = SL_n/SO_n$ in depth:
   (a) Show that if $p = gSO_n$ then the geodesics through p are $\gamma(t) = e^{tX} gSO_n = ge^{tY} SO_n$ for $gYg^{-1} = X$, with $X \in \mathfrak{p} = T_pM$ and Y a traceless symmetric matrix.
   (b) Show that the maximal flats in M correspond to (unordered) line decompositions in $\mathbb{R}^n$.
   (c) Let $p \in M$, and X a unit vector in $\mathfrak{p} = T_pM$. Consider

   $$F_{t,\lambda}(X) = \{v \in \mathbb{R}^n | \|e^{t(\lambda - X)} v\| \leq \|v\|\}.$$

   Extract a flag in $\mathbb{R}^n$ associated to the geodesic $e^{tX}p$, with in addition a positive real number associated to each subspace in the flag. This is called an *eigenvalue-flag pair*. Show that it depends only on the ideal point that X points to.

(d) Show that the eigenvalue-flag pair gotten above coincides with: Let E_i be the λ_i-eigenspace of X. Set the ith subspace of a flag to be the direct sum of the first i eigenspaces, ordered so that larger λ_i come first.

(e) Show that a Weyl chamber in M corresponds to a flag in $\mathbb{R}^n$.

(f) Show that the choice of a Weyl chamber in a flat F corresponds to choosing an ordering of the line decomposition.

(g) Show that any two ideal points lie in a common flat. When is that flat unique?

(h) Given two regular ideal points x, y in $M(\infty)$, represented as eigenvalue-flag pairs, find an easy way to calculate $\angle(x, y)$. What about singular points?

(i) What is the Weyl group? How does it act on a flat? How many Weyl chambers are in a given flat? How many different types of parabolic subgroups are there? (Say, non-conjugate).

(j) Find a geodesic γ with $F(\gamma)$ having a nontrivial non-Euclidean factor.