Symmetric Spaces
Exercises Day 8

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Assume M is a symmetric space of noncompact type and rank at least 2 and set $G = \mathrm{Isom}_0(M)$.

1. We show that any two Weyl faces lie in a common apartment.

   (a) Let γ be any geodesic of M and let w be a unit vector tangent to M at $\gamma(0)$ such that w and $\gamma'(0)$ are not collinear and span a 2-plane of zero sectional curvature. Then there exists a maximal flat F in M such that γ is contained in F and w is tangent to F. Hint: the sectional curvature $K(u,v) = \langle R(u,v)v, u\rangle = \langle R_v(u), u\rangle$ up to a positive scalar. The curvature transformation $R_v : T_pM \to T_pM$ becomes $R_X = (\mathrm{ad}\, X)^2$ on $\mathfrak{p}$.

   (b) A *Tits geodesic* is a continuous curve $\sigma : [a,b] \to M(\infty)$ that is parameterized proportional to arclength and is locally distance-minimizing. It is *minimal* if $d(\sigma(a), \sigma(b))$ equals the length of σ. Now let σ be a minimal Tits geodesic between antipodal ideal points. Show that there exists a maximal flat containing σ. Hint: find a point realizing the angle, which must be the corner of a flat triangular sector by an earlier proposition. Then use part 1.

   (c) Show that the same holds when the angle is strictly less than π, and moreover in this case the minimal Tits geodesic is unique.

2. Let W be a finite Coxeter group acting on a sphere S with no fixed points. Show that S has a natural simplicial complex structure.

3. Let τ_1, τ_2 be antipodal Weyl faces. Show that there exists a geodesic γ such that $P(\tau_1, \tau_2) = F(\gamma)$.

4. Let τ_1, τ_2 be antipodal Weyl faces. Show that $P_{\tau_1} \cap P_{\tau_2}$ act transitively on $P(\tau_1, \tau_2)$.