INTRO TO HOMOLOGICAL ALGEBRA AND SPECTRAL SEQUENCES MINICOURSE

RICHARD WONG

Abstract. The first three days of this course will be lectures on the basics of homological algebra with an eye towards computations. Topics include diagram chasing, working with chain complexes, computing homology and cohomology, using the Kunneth and universal coefficient theorems, and computing Ext and Tor groups.

On Thursday and Friday, we will discuss spectral sequences, focusing in particular on the Kunneth and Serre spectral sequences, which are also very useful for calculating homology / cohomology of topological spaces. There will be a problem session from 1-2pm.

1. Monday

Let R be a commutative ring. We will develop homological algebra in the category of R-modules. That is, we will construct the category $\mathsf{Ch}(R)$ of chain complexes of R-modules.

1.1. Objects

Definition 1.1. A chain complex of R-modules, X, is a sequence of maps (called differentials) of R-modules

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

such that $d_n \circ d_{n+1} = 0$ for all n.

Definition 1.2. A cochain complex of R-modules, Y, is a sequence of maps of R-modules

$$\cdots \to Y^{n-1} \xrightarrow{d^{n-1}} Y^n \xrightarrow{d^n} Y^{n+1} \to \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all n.

Note that chain complexes have homological grading: the differential lowers degree, while cochain complexes have cohomological grading: the differential raises degree.

The usual convention is that chain complexes that are bounded above: $X_n = 0$ for $n < 0$, and cochain complexes that are bounded below: $Y^n = 0$ for $n < 0$. Without the boundedness restrictions, the notions are equivalent.

Example 1.3 Given an R-module M, we can regard it as a chain complex concentrated in degree 0.

Example 1.4 Given an R-module homomorphism $f : M \to N$, we can regard it as a chain complex concentrated in degrees 0 and 1.

Example 1.5 We can define a chain complex I to be the chain complex with $R^{\oplus 2}$ in degree 0, generated by elements $[0]$ and $[1]$, and a R in degree 1 with generator $[I]$, and 0 everywhere else. The only non-trivial differential sends $d([I]) \mapsto [0] - [1]$.

This is the "cylinder object" in $\mathsf{Ch}(R)$. It plays the analogous role that the interval $[0, 1]$ does in the category Top_*.

An element in the kernel of d_k is called a cycle, and an element in the image of d_{k+1} is called a boundary. Two cycles f, g are "homologous" if $f - g$ is a boundary. Note that $\ker d_k$ and $\mathrm{im} d_{k+1}$ are submodules ofX_k, and moreover we have that $\mathrm{im} d_{k+1} \subseteq \ker d_k \subseteq X_k$ thanks to our condition on the differentials.

Definition 1.6. We define the kth homology of a chain complex X to be the following abelian group:

$$H_k(X) = \ker d_k / \mathrm{im} d_{k+1}$$

Similarly, we define the kth cohomology of a cochain complex Y to be the following abelian group:

$$H^k(Y) = \ker d^k / \mathrm{im} d^{k+1}$$

Remember that the usual convention is that chain complexes that are bounded above: $X_n = 0$ for $n < 0$, and cochain complexes that are bounded below: $Y^n = 0$ for $n < 0$. This is because we prefer to have homology (respectively cohomology) to live in positive degrees.

So again in the world of complexes, the difference in homology and cohomology is whether we are looking at chain vs. cochain complexes.

Example 1.7 Let $R = \mathbb{Z}$. Then R-modules are the same as abelian groups. Suppose that G is a finitely generated abelian group. Then we can construct a chain complex C such that $H_0(C) = G$

Write G in terms of generators and relations, and let $C = 0 \to \mathbb{Z}^n \to \mathbb{Z}^m \to 0$.

Example 1.8 Let $f : M \to N$ be a morphism of R modules, considered as a chain complex C. Then $H_0(C) = \mathrm{coker}(f)$ and $H_1(C) = \ker(f)$.

Example 1.9 A chain complex that has 0 homology is said to be exact.

1.2. **Morphisms**

Definition 1.10. A map of chain complexes $f : X \to X'$ is a sequence of maps of R-modules $f_n : X_n \to X'_n$ that commute with the differentials for all n. That is, we have the following commutative diagram for all n.

$$\begin{array}{ccc} X_n & \xrightarrow{d_n} & X_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ X'_n & \xrightarrow{d'_n} & X'_{n-1} \end{array}$$

Therefore, we have that $f_k(\mathrm{im} d_{k+1}) \subseteq \mathrm{im} d'_{k+1}$ and $f_k(\ker d_k) \subseteq \ker d'_k$, and hence f induces a map on homology: $f_* : H_k(X) \to H_k(X')$.

As always, we don't care about all maps between chain complexes, we only care about maps up to some equivalence relation. The question is: which equivalence relation should we take?

Definition 1.11. A chain homotopy h between maps $f, g : X \to X'$ is a sequence of maps of R-modules $h_n : X_n \to X'_{n+1}$ such that we have the following commutative diagram for all n.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots \\ & & f \downarrow\downarrow g & \swarrow h & f \downarrow\downarrow g & \swarrow h & f \downarrow\downarrow g & & \\ \cdots & \longrightarrow & X'_{n+1} & \longrightarrow & X'_n & \longrightarrow & X'_{n-1} & \longrightarrow & \cdots \end{array}$$

Lemma 1.12. *Chain homotopic maps induce the same homomorphism of homology groups.*

Definition 1.13. Two complexes X, X' are said to be chain homotopic if there are chain maps $f : X \to X'$, $g : X' \to X$ such that there are chain homotopies $s : g \circ f \to id_X$ and $t : f \circ g \to id_{X'}$.

f and g are said to be chain homotopy equivalent.

Lemma 1.14. *Chain homotopy equivalent maps induce isomorphisms on homology.*

However, this turns out to not be quite the right equivalence relation, as we will see in the exercises later today. The notion of chain homotopy is a bit too strict.

When we consider chain complexes, we usually care about their homology groups. Therefore, we would like a notion of equivalence that allows us to replace one chain complex with another one with the same homology. This leads us to the notion of quasi-isomorphism.

Definition 1.15. A map of chain complexes $f : X \to X'$ is said to be a quasi-isomorphism if for each k, the induced morphism on homology $f_* : H_k(X) \to H_k(X')$ is an isomorphism.

The two chain complexes are said to be quasi-isomorphic.

Remark 1.16. Quasi-isomorphism is not an equivalence relation: it is not symmetric, so we have to formally add inverse maps through a process called localization.

Remark 1.17. It is not enough to have two complexes with the same homology, we require also a map between them that induces an isomorphism on homology.

Definition 1.18. A chain complex quasi-isomorphic to the 0 chain complex is said to be acyclic.

1.3. Constructions

Definition 1.19. Given two chain complex of R-modules X and Y, then we can form their tensor product (over R), denoted $X \otimes Y$, in the following way:

$$(X \otimes Y)_n = \sum_{p+q=n} X_p \otimes_R Y_q$$

and the differential d is given by

$$d(x \otimes y) = d_X(x) \otimes y + (-1)^p x \otimes d_Y(y)$$

for $x \in X_p$ and $y \in Y_q$. One can check that this sign convention ensures that $d \circ d = 0$.

Example 1.20 Given a chain complex of R-modules X and an R-modules M, we can then form the chain complex $X \otimes_R M$. This has in the nth degree the R-module $X_n \otimes M$, and differential $d_X \otimes id_M$.

The homology $H_n(X \otimes_R M)$ is called the homology of X with coefficients in M, sometimes denoted $H_n(X; M)$.

Definition 1.21. Given a chain complex of R-modules X_*, we define the dual cochain complex $D(X)$ in the following way:

$$D(X)^n = \mathrm{Hom}_R(X_n, R)$$

$$d^n = (-1)^n \mathrm{Hom}_R(d_{n+1}, id)$$

.

On elements, given an R-module map $f : X_n \to R$ and an element $x \in X_{n+1}$, we have $(d^n f)(x) = (-1)^n f(d_n(x))$.

Example 1.22 Note that in general, $H^n(D(X)) \ncong \mathrm{Hom}(H_n(X), \mathbb{Z})$. (destroys torsion information)

In general, given an R-module M, we can define in the same way a cochain complex $\mathrm{Hom}(X, M)$, and we denote the cohomology $H^n(\mathrm{Hom}(X, M))$ by $H^n(X; M)$.

Definition 1.23. Given a chain map $f : X \to Y$, the mapping cone of f is the complex $C(f)$ defined by

$$C(f)_n = X_{n-1} \oplus Y_n$$

and differential

$$d(x, y) = (-d_X(x), d_Y(y) - f(x))$$

if f is a cofibration(inclusion), then $C(f)$ is the same thing as the quotient. It has the universal property of being a pushout over a point, and gives a LES in homology. (detects quasi-isomorphisms)

Example 1.24 We construct cellular homology by constructing the cellular chain complex $CW_\bullet(X)$ for a CW complex X..

We take $CW_n(X)$ to be the free abelian group $H_n(X_n, X_{n-1})$, which has generators the n-cells of X.

The boundary map sends a cell $d_n(e_n^\alpha) = \sum \deg(\chi_n^{\alpha\beta} e_{n-1}^\beta)$, where $\chi_n^{\alpha\beta}$ is the map induced by the attaching map $S^{n-1} \to X_{n-1}$ composed with the quotient map $X_{n-1}/(X_{n-1} \setminus e_{n-1}^\beta) \cong S^{n-1}$.

Example 1.25 We construct singular homology by constructing the singular complex $C_\bullet(X)$.

We take $C_n(X)$ to be the free abelian group generates by all singular n-simplices on a topological space X. That is, maps $\sigma_n : \Delta^n \to X$.

The boundary map is restriction onto the faces of the simplex:

$$\delta_n(\sigma_n) = \sum_{k=0}^{n} (-1)^k [p_0, \cdots, p_{k-1}, p_{k+1}, \cdots, p_n]$$

Example 1.26 We obtain $H_n(X;G)$ for G an abelian group by taking the homology of $C_\bullet(X)\otimes G$.

Example 1.27 We obtain $H^n(X;G)$ for G an abelian group by taking the dual cochain complex of $C_\bullet$.

2. Monday Exercises

Exercise 2.1. Prove the five lemma. In the diagram below, if the rows are exact, m and p are isomorphisms, l is an epimorphism, and q a monomorphism, then n is an isomorphism.

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{j'} & E' \end{array}$$

Exercise 2.2. Prove the splitting lemma: For any short exact sequence of R-modules $0\to A\xrightarrow{f} B\xrightarrow{g} C\to 0$, then the following are equivalent:

(1) There exists a map $u : B\to A$ such that $u\circ f = id_A$.
(2) There exists a map $v : C\to B$ such that $g\circ v = id_C$.
(3) $B\simeq A\oplus C$.

Exercise 2.3. Show that the splitting lemma does not hold in the category of groups. (**Hint:** Consider the SES of groups $0\to A_3\xrightarrow{i} S_3\xrightarrow{det}\mathbb{Z}/2\to 0$)

Exercise 2.4. Watch the 1980 film, "It's My Turn", an American romantic comedy-drama film starring Jill Clayburgh, Michael Douglas and Charles Grodin.

Then prove the snake lemma.

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

In the diagram above, if the rows are exact, then there is an exact sequence

$$\ker(f)\to\ker(g)\to\ker(h)\to\operatorname{coker}(f)\to\operatorname{coker}(g)\to\operatorname{coker}(h)$$

Exercise 2.5. Show that a short exact sequence $0\to A\to B\to C$ of chain complexes of R-modules givess rise to a long exact sequence of homology groups

$$\cdots\to H_k(A)\to H_k(B)\to H_k(C)\to H_{k-1}(A)\to H_{k-1}(B)\to H_{k-1}(C)\to H_{k-2}(A)\to\cdots$$

Exercise 2.6. Given a cochain complex, you can form its dual chain complex. Show that a chain complex is not necessarily quasi-isomorphic to its double dual (the dual of the dual chain complex.)

Hint: Consider the chain complex of abelian groups

$$\cdots\to 0\to\mathbb{Z}/2\to 0\cdots$$

Exercise 2.7. Show that if k is a field, and X is a chain complex of k-modules, then

$$H^*(D(X))\cong D(H^*(X))$$

(Hint: Show that $D(-) := \operatorname{Hom}(-,k)$ is an exact functor. That is, it preserves kernels and cokernels. Then show that exact functors commute with (co)homology.

Exercise 2.8. If A is a bounded chain complex of finitely generated abelian groups (that is, all but finitely many A_n are 0, and all the A_n are finitely generated), then we can define the Euler characteristic

$$\chi(A)=\sum(-1)^n\operatorname{rank}(A_n)$$

Show that Euler characteristic can also be computed on the level of homology, that is

$$\chi(A)=\sum(-1)^k\operatorname{rank}(H_k(A))$$

Exercise 2.9. Show that there exists a chain homotopy equivalence $f : X \to X'$ where $f_n : X_n \to X'_n$ is not an isomorphism by considering the chain complexes

$$\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \cdots$$
$$\cdots \to 0 \to 0 \to 0 \to 0 \cdots$$

Exercise 2.10. Show that quasi-isomorphism is not symmetric by considering the chain complexes of abelian groups

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \cdots$$
$$\cdots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \cdots$$

This also gives an example of a quasi-isomorphism that is not a chain homotopy equivalence.

Exercise 2.11. Let $R = k[x, y]$. Show that the chain complexes of R-modules

$$0 \to R \oplus R \xrightarrow{h} R \to 0$$
$$0 \to R \xrightarrow{0} k \to 0$$

have the same homology but are not quasi-isomorphic. The map h is defined by

$$h(f(x, y), g(x, y)) = xf(x, y) + yg(x, y)$$

Exercise 2.12. Show that the category of chain complexes of R-modules up to quasi-isomorphism (that is, you have the same objects of $\mathsf{Ch}(R)$, but you invert all the quasi-isomorphisms) is not abelian, but is instead triangulated. This category is called the derived category of R.

(Hint: certain kernels/cokernels do not necessarily exist).

3. Tuesday

3.1. Universal coefficients in homology

Today we will work exclusively over a PID R.

We will cover theorems that allow us to compute homology and cohomology groups of a complex (usually with coefficients) given knowledge of other homology/cohomology groups. In particular, we will compute $H_*(X; M)$ given $H_*(X)$, $H_*(X \otimes Y)$ given $H_*(X) \otimes_R H_*(Y)$, and $H^*(X; M)$ given $H_*(X)$.

First, we have a map $\alpha : H_*(X) \otimes H_*(X) \to H_*(X \otimes Y)$ defined by

$$\alpha([x] \otimes [y]) = [x \otimes y]$$

Theorem 3.1 (Universal Coefficient Theorem for Homology)**.** *Let M be a module over a principal ideal domain R. Let C be a chain complex of flat (read:free) R-modules. We would like to compute the homology groups $H_k(C \otimes_R M)$ of the chain complex $C \otimes_R M$.*

For each k, there is a natural short exact sequence of R-modules

$$0 \to H_k(C) \otimes_R M \xrightarrow{\alpha} H_k(C \otimes_R M) \to \operatorname{Tor}_1^R(H_{k-1}(C), M) \to 0$$

Corollary 3.2 (Universal Coefficient Theorem for Integral Homology)**.** Let X be a topological space and G be an abelian group, and suppose we know $H_*(X) := H_*(X; \mathbb{Z})$. Then one can compute $H_k(X; G)$ for each k with the natural short exact sequence of abelian groups

$$0 \to H_k(X) \otimes G \xrightarrow{\alpha} H_k(X; G) \to \operatorname{Tor}(H_{k-1}(X), G) \to 0$$

Remark 3.3. In fact, these short exact sequences split, although the splitting is not natural, as we will see in the exercises below.

How do we compute the group $\operatorname{Tor}_1^R(H_{k-1}(C), M)$? Given an R-module M, we can construct a short exact sequence

$$0 \to F_1 \to F_0 \to M \to 0$$

where F_0 and F_1 are free modules. We choose $F_0 \to M$ to be a surjection, and F_1 to be the kernel of that surjection. Since R is a PID, and F_1 is a submodule of a free module over R, it is also free.

sketch idea: M finitely generated, have the theorem describing it. Take F_0 to be free on cyclic generators. proof of submodule, induct on dimension of free module (submodule of R is ideal), $A \to A(a)$ is injective.

Induction, split $A^{n+1} = A \oplus A^n$, project on to A^n part. Then M surjection onto free module $\pi(M)$, hence $M \cong \pi(M) \oplus ker\pi$

Then we tensor with N and obtain an exact sequence

$$0 \to \mathrm{Tor}_1^R(M,N) \to F_1 \otimes N \ \to F_0 \otimes N \to M \otimes N \to 0$$

This is special to the case that R is a PID. You will get more Tor groups over other rings. This measures the failure of the functor $(-) \otimes N$ to be exact.

Proposition 3.4. We have the following properties of Tor, Hatcher-style.

(1) $\mathrm{Tor}_1^R(\oplus_i M_i, N) \cong \oplus_i \mathrm{Tor}_1^R(M_i, N)$
(2) $\mathrm{Tor}_1^R(M,N) = 0$ if M (or N) is free.
(3) $\mathrm{Tor}(\mathbb{Z}/n, A) = ker(A \xrightarrow{\cdot n} A)$ for A an abelian group.
(4) For each short exact sequence $0 \to A \to B \to C \to 0$, we have a natural exact sequence

$$0 \to \mathrm{Tor}_1^R(M,A) \to \mathrm{Tor}_1^R(M,B) \to \mathrm{Tor}_1^R(M,C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$

(5) $\mathrm{Tor}_1^R(M,N) \cong \mathrm{Tor}_1^R(N,M)$.

3.2. Künneth Theorem

We can in fact ask a more general question:

Theorem 3.5 (Künneth)**.** *Let R be a principal ideal domain, Let C be a chain complex of flat (read:free) R-modules, and D be any chain complex. We would like to compute the homology groups $H_k(C \otimes_R D)$.*

For each k, there is a natural short exact sequence of abelian groups

$$0 \to \bigoplus_{i+j=k} H_i(C) \otimes_R H_j(D) \to H_k(C \otimes_R D) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0$$

Corollary 3.6 (Künneth Formula)**.** Let X and Y be topological spaces and F be a field. Then for each integer k we have a natural isomorphism

$$\bigoplus_{i+j=k} H_i(X;F) \otimes H_j(Y;F)) \to H_k(X \times Y; F)$$

For a general ring R, we have a Kunneth spectral sequence with E_2 page $\mathrm{Tor}_p^R(H_{q_1}, H_{q_2}) \Rightarrow H_{p+q}(X \times Y)$. We will discuss this in later lectures.

Theorem 3.7 (Eilenberg-Zilber)**.** *Given two topological spaces X and Y, we have an isomorphism of singular chain complexes (over R)*

$$C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$$

3.3. Universal coefficients in cohomology

First, we define a map $\alpha : H^k(C;M) \to \mathrm{Hom}_R(H_k(C), M)$ which is defined for each k by $\alpha([f])([x]) = f(x)$ for a cohomology class represented by a cocycle $f : X_n \to M$ and cycle $x \in H_n(X)$.

Theorem 3.8 (Universal Coefficient Theorem for Cohomology)**.** *Let M be a module over a principal ideal domain R. Let C be a chain complex of free R-modules. We would like to compute the cohomology groups $H^k(C;M)$ of the cochain complex* $\mathrm{Hom}(C_n, M)$*.*

For each k, there is a natural short exact sequence of R-modules

$$0 \to \mathrm{Ext}_R^1(H_{k-1}(C), M) \to H^k(C;M) \xrightarrow{\alpha} \mathrm{Hom}_R(H_k(C), M) \to 0$$

Corollary 3.9 (Universal Coefficient Theorem for Integral Cohomology)**.** Let X be a topological space and G be an abelian group, and suppose we know $H_*(X) := H_*(X;\mathbb{Z})$. Then one can compute $H^k(X;G)$ for each k with the natural short exact sequence of abelian groups

$$0 \to \mathrm{Ext}(H_{k-1}(X), G) \to H^k(X;G) \to \mathrm{Hom}(H_k(X), G) \to 0$$

Remark 3.10. In fact, these short exact sequences again split, although the splitting is not natural, as we will see in the exercises below.

How do we compute the group $\mathrm{Ext}_1^R(H_{k-1}(C), M)$? Given an R-module M, we can construct a short exact sequence

$$0 \to F_1 \to F_0 \to H_{k-1}(C) \to 0$$

where F_0 and F_1 are free modules.

Then we apply $\mathrm{Hom}(-, N)$ to get the dual cochain complex, and we obtain an exact sequence

$$0 \to \mathrm{Hom}(H_{k-1}(C), N) \to \mathrm{Hom}(F_0, M) \to \mathrm{Hom}(F_1, M) \to \mathrm{Ext}_1^R(H_{k-1}(C), M) \to 0$$

Again, this is special to the case that R is a PID. You will get more Ext groups over other rings. This measures the failure of the functor $\mathrm{Hom}(-, N)$ to be exact.

We can also compute Ext by applying $\mathrm{Hom}(H_{k-1}(C), -)$ to the SES $0 \to F_1' \to F_0' \to M \to 0$.

Proposition 3.11. We have the following properties of Ext, Hatcher-style.

1. $\mathrm{Ext}_R^1(\oplus_i M_i, N) \cong \oplus_i \mathrm{Ext}_R^1(M_i, N)$
2. $\mathrm{Ext}_R^1(M, N) = 0$ if M is free (or in general, projective).
3. $\mathrm{Ext}(\mathbb{Z}/n, A) = A/nA$ for A an abelian group.
4. For each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence
$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to \mathrm{Ext}_R^1(C, N) \to \mathrm{Ext}_R^1(B, N) \to \mathrm{Ext}_R^1(A, N) \to 0$$
5. Similarly, for each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence
$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \mathrm{Ext}_R^1(M, A) \to \mathrm{Ext}_R^1(M, B) \to \mathrm{Ext}_R^1(M, C) \to 0$$

4. Tuesday Exercises

Exercise 4.1. We saw the Kunneth/Universal Coefficient theorems about how to compute $H_*(X; M)$ given $H_*(X)$, $H_*(X \otimes Y)$ given $H_*(X) \otimes_R H_*(Y)$, and $H^*(X; M)$ given $H_*(X)$.

State and prove the analogous theorems Kunneth/Universal Coefficient theorems for cohomology. (computing $H^*(X; M)$ given $H^*(X)$, $H^*(X \otimes Y)$ given $H^*(X) \otimes_R H^*(Y)$, and $H_*(X; M)$ given $H^*(X)$)

Exercise 4.2. Prove the following properties of Tor and Ext.

1. $\mathrm{Tor}_1^R(\oplus_i M_i, N) \cong \oplus_i \mathrm{Tor}_1^R(M_i, N)$
2. $\mathrm{Tor}_1^R(M, N) = 0$ if M (or N) is free.
3. $\mathrm{Tor}(\mathbb{Z}/n, A) = ker(A \xrightarrow{\cdot n} A)$ for A an abelian group.
4. For each short exact sequence $0 \to A \to B \to C \to 0$, we have a natural exact sequence
$$0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$
5. $\mathrm{Tor}_1^R(M, N) \cong \mathrm{Tor}_1^R(N, M)$.

(1) $\mathrm{Ext}_1^R(\oplus_i M_i, N) \cong \oplus_i \mathrm{Tor}(M_i, N)$
(2) $\mathrm{Ext}_1^R(M, N) = 0$ if M is free (or in general, projective).
(3) $\mathrm{Ext}(\mathbb{Z}/n, A) = A/nA$ for A an abelian group.
(4) For each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence
$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to \mathrm{Ext}_1^R(C, N) \to \mathrm{Ext}_1^R(B, N) \to \mathrm{Ext}_1^R(A, N) \to 0$$
(5) Similarly, for each short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have a natural exact sequence
$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \mathrm{Ext}_1^R(M, A) \to \mathrm{Ext}_1^R(M, B) \to \mathrm{Ext}_1^R(M, C) \to 0$$

Exercise 4.3. Suppose that X is a topological space such that $H_k(X)$ is free and finitely generated for all k. Prove that $H^k(X) \cong H_k(X)$.

(a) Compute $H^k(\mathbb{C}P^\infty)$ for all n.
(b) Compute $H^k(S^n)$ for all k, n.

Exercise 4.4. Suppose that R is a field. Then show that $H_*(X) \otimes_R M \cong H_*(X; M)$, and $H^*(X; M) \cong \mathrm{Hom}(H_*(X), M)$.

Exercise 4.5. Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 4.6. Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/m)$ for all k and for any odd integer $m \geq 3$.

Exercise 4.7. Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z})$ for all k.

Exercise 4.8. Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 4.9. Compute $H^k(\mathbb{R}P^\infty; \mathbb{Z})$ for all k.

Exercise 4.10. Compute $H^k(\mathbb{R}P^\infty; \mathbb{Z}/2)$ for all k.

Exercise 4.11. Let G be an abelian group. Let $M(G, n)$ be a Moore space, that is to say it satisfies

$$\widetilde{H_k}(M(G,n)) = \begin{cases} G & k = n \\ 0 & else \end{cases}$$

(a) Construct a CW complex $M(\mathbb{Z}/m, n)$ having 3 cells.
(b) Consider the map $f : M(\mathbb{Z}/m, n) \to S^{n+1}$ that collapses the n-skeleton. Use this map to show that the splitting in the Universal Coefficient Theorem is not natural.
(c) Consider the map $f \times id : M(\mathbb{Z}/m, n) \times M(\mathbb{Z}/m, n) \to S^{n+1} \times M(\mathbb{Z}/m, n)$. Use this map to show that the splitting in the Künneth Theorem is not natural.

5. Wednesday

Today we talk about Tor and Ext over general rings, not just PIDs.

First we discuss flat, projective, injective, flat modules:

Definition 5.1. A flat module M over a ring R is a module such that $- \otimes_R M$ preserves exact sequences. (always right exact)

Lemma 5.2. *Free modules are flat.*

Theorem 5.3. *The following are equivalent definitions of a projective module:*

(i)
$$\begin{array}{ccc} & & M \\ & \overset{h}{\nearrow} & \big\downarrow f \\ P & \xrightarrow[g]{} & N \end{array}$$

(ii) every short exact sequence of the form $0 \to A \to B \to P \to 0$ *splits.*
(iii) There is a free module F *and a module* Q *such that* $P \oplus Q = F$*.*
(iv) $\mathrm{Hom}(P, -)$ *is an exact functor. (always left exact, but right exact if* P *is projective.)*

Lemma 5.4. *Any projective module is flat.*

Free modules are flat, direct sums and summands of flat modules are flat. Projectives are direct summands of free modules.

Definition 5.5. A projective resolution of an R module M is a chain complex such that the following complex is exact

$$\cdots X_{i+1} \to X_i \xrightarrow{d_i} \cdots X_0 \xrightarrow{\varepsilon}\to M \to 0$$

Proposition 5.6. Every R-module has a projective (free!) resolution. This is unique up to chain homotopy (and hence quasi-isomorphism) via property of projective modules.

We can now define Tor:

Definition 5.7. Let M, N be R-modules, and choose a projective (flat) resolution $X \to M$. Then

$$\mathrm{Tor}_*^R(M, N) := H_*(X \otimes_R N)$$

Proposition 5.8. We have the following properties of Tor :

(1) $\mathrm{Tor}_n^R(M, N) = 0$ for $n < 0$.
(2) $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$.
(3) we can also resolve N instead to compute Tor.

Tor are the left derived functors of $\otimes M$, which is right exact. This is what we mean when we say that Tor measures the failure of $- \otimes M$ to be exact.

Example 5.9 For R a PID, we see that $\mathrm{Tor}_n^R(M, N) = 0$ for $n > 0$.

Dually, we can define injective modules.

Theorem 5.10. *The following are equivalent definitions of an injective module:*

(i)
$$\begin{array}{ccc} & & M \\ & \swarrow^{h} & \big\downarrow f \\ I & \xleftarrow[g]{} & N \end{array} \quad g \text{ dashed}$$

(ii) every short exact sequence of the form $0 \to I \to A \to B \to 0$ *splits.*
(iii) if submodule, then direct summand
(iv) $\mathrm{Hom}(-, I)$ *is an exact functor. (always left exact, but right exact if* I *is injective.)*

Definition 5.11. An injective resolution of an R module N is a cochain complex Y such that the following complex is exact

$$0 \to N \xrightarrow{\eta} Y^0 \cdots Y^i \to Y^{i+1} \to \cdots$$

Proposition 5.12. Every R-module embeds as a submodule of an injective R-module.
$N \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, N) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, D)$
for some homomorphism $N \to D$ of abelian groups, where D is divisible (aka an injective $\mathbb{Z}$-module, Baer's criterion).

Proposition 5.13. Every R-module has an injective resolution. This is unique up to chain homotopy (and hence quasi-isomorphism) via property of injective modules.
This is not functorial!

We can now define Ext:

Definition 5.14. Let M, N be R-modules, and choose a projective (flat) resolution $X \to M$. Then

$$\mathrm{Ext}_R^*(M, N) := H_*(\mathrm{Hom}(X, N))$$

Alternatively, choose an injective resolution $N \to Y$. Then

$$\mathrm{Ext}_R^*(M, N) := H_*(\mathrm{Hom}(M, Y))$$

Proposition 5.15. We have the following properties of Ext :

(1) $\mathrm{Ext}_R^n(M, N) = 0$ for $n < 0$.
(2) $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}(M, N)$.
(3) $\mathrm{Ext}_R^n(M, N) = 0$ for M projective or N injective.

Ext are the right derived functors of $\mathrm{Hom}(-, N)$ or $\mathrm{Hom}(M, -)$, which are left exact.

Example 5.16 For R a PID, we see that $\mathrm{Ext}_R^n(M, N) = 0$ for $n > 0$.

Definition 5.17. In particular, this is very interesting when we take $R = kG$, the group ring of a (finite) group G. k is a commutative ring, usually a Dedekind domain. We will think of k as a field or as $\mathbb{Z}$. Let M be a kG-module, and we then define group cohomology to be the right derived functors of $\mathrm{Hom}(k, -)$, where k is the trivial kG-module.

$$H^n(G, M) := \mathrm{Ext}_{kG}^n(k, M)$$

We can also define group homology in the same way:

$$H_n(G, M) := \mathrm{Tor}_n^{kG}(k, M)$$

We can compute these groups by taking a projective resolution of $\mathbb{Z}$: In this case, we may use the bar construction:

Example 5.18 We have a resolution of $\mathbb{Z}$ given by the following complex, where $F_n = \bigoplus_{\underline{g} \in G^n} \mathbb{Z}G\{\underline{g}\}$, the free $\mathbb{Z}G$ module on G^n. This is the same as $\mathbb{Z}[G^{n+1}]$ with diagonal action (free module with basis $(1, g_1, g_1g_2, \cdots, g_1g_2\cdots g_n)$. differential is alternating sum omit ith entry.

The differential is given by a map $\Delta : G^n \to F_{n-1}$; the augmentation map $\sum_{g\in G} c_g[g] \mapsto \sum_{g \in G} c_g$. $d_1(\sum_{g\in G} x_i[g_i]) = \sum x([g_i] - [1])$. The rest of the maps are given by $d_k(\sum_i x_i(g_i) = \sum_i x_i \Delta(g_i)$, $\Delta_j(g_1, \cdots, g_j) = [g_1](g_1, \cdots g_j) + alternatingsum, multiply 2$

Example 5.19 This is big and very messy, and sometimes we can find a nice minimal resolution, such as when we have a cyclic group.

$$\mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

$\cdot(g-1)$ is first map, followed by $\sum_{i=0}^{N-1} g^i$, etc. two step resolution.

5.1. **Other resolutions**

Example 5.20 Koszul resolution for regular sequence $R/(s_1, \cdots, s_n)$

6. Wednesday Exercises

Exercise 6.1. Where does the term Ext come from? Show that $\mathrm{Ext}^1_R(M, N)$ is isomorphic to the group of extensions up to equivalence of M by N, with addition given by the Baer sum:

Definition 6.2. If M and N are R-modules, then we say an extension of M by N is a SES of R-modules:

$$0 \to N \to E \to M \to 0$$

Definition 6.3. Two extensions $0 \to N \to E \to M \to 0$ and $0 \to N \to E' \to M \to 0$ are equivalent if there is a commutative diagram, middle is isomorphism

Proposition 6.4. The set of equivalence classes of extensions has a group structure coming from Baer sum. Given two extensions $0 \to N \to E \to M \to 0$ and $0 \to N \to E' \to M \to 0$, form the SES $0 \to N \oplus N \to E \oplus E' \to M \oplus M \to 0$. Pull back along the diagonal map $M \to M \oplus M$, and then push out along the sum map $A \oplus A \to A$ to obtain the Baer sum $0 \to N \to E'' \to M \to 0$

Exercise 6.5. Generalize the previous exercise to show that $\mathrm{Ext}^n_R(M, N)$ is isomorphic to the group of extensions of M by N of length n up to equivalence.

Definition 6.6. If M and N are R-modules, then we say an extension of M by N of length n is an exact sequence of R-modules:

$$0 \to N \to E_{n-1} \to \cdots E_0 \to M \to 0$$

Exercise 6.7. Show that we have an associative and unital map called the Yoneda product:

$$\mathrm{Ext}^n_R(N, P) \otimes \mathrm{Ext}^m_R(M, N) \to \mathrm{Ext}^{m+n}_R(M, P)$$

by considering the isomorphism bewteen $\mathrm{Ext}^n_R(M, N)$ and the group of extensions of M by N of length n up to equivalence.

Exercise 6.8. Let M be a $\mathbb{Z}G$-module. Show that $H^0(G; M) = M^G$, the G-fixed points of M. Also show that $H_0(G; M) = M_G$, the coinvariants of M. In other words, M_G is the quotient of M by the submodule generated by elements of the form $g \cdot m - m$.

Exercise 6.9. Show that if k is a field of characteristic p, and G is $\mathbb{Z}/p$, then

$$H^*((G)^n, k) = \begin{cases} \mathbb{F}_p[x_1, \cdots, x_n] & |x_i| = 1,\ p = 2 \\ \Lambda[x_1, \dots x_n] \otimes \mathbb{F}_p[y_1, \cdots, y_n] & |x_i| = 1, |y_i| = 2,\ p \neq 2 \end{cases}$$

7. Thursday

Today we will begin discussing spectral sequences, an extremely useful computational tool.

We will first discuss the spectral sequence associated to a double complex, and then see how one can obtain the Kunneth and Universal coefficient spectral sequences.

Definition 7.1. A double chain complex is a collection of R-modules with rightward and downward differentials that commute.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{i+1,j+1} & \xrightarrow{d_H} & X_{i,j+1} & \xrightarrow{d_H} & X_{i-1,j+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
\cdots & \longrightarrow & X_{i+1,j} & \xrightarrow{d_H} & X_{i,j} & \xrightarrow{d_H} & X_{i-1,j} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
\cdots & \longrightarrow & X_{i+1,j-1} & \xrightarrow{d_H} & X_{i,j-1} & \xrightarrow{d_H} & X_{i-1,j-1} & \longrightarrow & \cdots
\end{array}$$

Definition 7.2. The total complex of a double chain complex X is a chain complex defined by

$$(\mathrm{Tot}(X))_n = \bigoplus_{i+j=n} X_{i,j}$$

with differential

$$d(X_{i,j}) := d_V(X_{i,j}) + (-1)^j d_H(X_{i,j})$$

Example 7.3 Let $f : X \to Y$ be a map of chain complexes. Consider the double complex given by

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{i+1} & \longrightarrow & X_i & \longrightarrow & X_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \longrightarrow & Y_{i+1} & \longrightarrow & X_i & \longrightarrow & X_{i-1} & \longrightarrow & \cdots
\end{array}$$

Then the total complex is chain homotopic to the mapping cone of f, $C(f)$.

Example 7.4 Let M be a module over a ring R. Let C be a chain complex of free R-modules. We would like to compute the cohomology groups $H^k(C; M)$ of the cochain complex $\mathrm{Hom}(C_p, M)$.

Take an injective resolution of $M \to I$ and then $\mathrm{Hom}(C_p, I_q)$ is a double complex.

Example 7.5 Let M and N be a module over a ring R, with projective resolutions of P and Q respectively. We can form a double complex $P \otimes Q$.

Our goal is often to compute the homology of the total complex.

The naive thing to do is to compute $H_H(H_V(X))$. But notice that we can also compute $H_V(H_H(X))$, and we have seen that these two things are not necessarily the same. The question is then, how are these things related, and how are they related to the homology of the total complex?

Example 7.6 Let M be a module over a ring R. Let C be a chain complex of free R-modules, and let $M \to I$ be an injective resolution and then $\mathrm{Hom}(C_p, I_q)$ is a double complex.

If we take the horizontal homology, since I_q is injective, then $\mathrm{Hom}(-, I_q)$ is exact. Therefore, it commutes with taking the homology, so

$$H_H(\mathrm{Hom}(C_p, I_q)) \cong \mathrm{Hom}(H_p(C), I_q)$$

Therefore, then when we take the vertical homology, we obtain

$$H_V(H_H(\mathrm{Hom}(C_p, I_q))) \cong H_V(\mathrm{Hom}(H_p(C), I_q)) := \mathrm{Ext}^q(H_p(C), M)$$

On the other hand, if we take the vertical homology first, since Cp is free (or projective), we observe that therefore $\mathrm{Hom}(C_p, -)$ is exact, and so we have that

$$H_V(\mathrm{Hom}(C_p, I_q)) = \begin{cases} 0 & q > 0 \\ \mathrm{Hom}(C_p, G) & q = 0 \end{cases}$$

Therefore, when we take the horizontal homology, we obtain

$$H_H(H_V(\mathrm{Hom}(C_p, I_q))) = \begin{cases} 0 & q > 0 \\ H^p(C; G) & q = 0 \end{cases}$$

Example 7.7 Let M and N be a module over a ring R, with projective resolutions of P and Q respectively. We can form a double complex $P_p \otimes Q_q$.

If we take the horizontal homology, since Q_q is projective, it is also flat, and hence $\otimes Q_q$ is exact. Hence it commutes with taking the homology, so

$$H_H(P \otimes Q) = \begin{cases} 0 & p > 0 \\ M \otimes Q_q & p = 0 \end{cases}$$

Then when we take the vertical homology, we obtain

$$H_V(H_H(P \otimes Q)) = \begin{cases} 0 & p > 0 \\ \mathrm{Tor}_q(M, N) & p = 0 \end{cases}$$

Similarly, if we take the vertical homology, since P_p is projective, it is also flat, and hence $\otimes P_p$ is exact. Hence

$$H_V(P \otimes Q) = \begin{cases} 0 & q > 0 \\ P_p \otimes N & q = 0 \end{cases}$$

Then when we take the horizontal homology, we obtain

$$H_H(H_V(P \otimes Q)) = \begin{cases} 0 & q > 0 \\ \mathrm{Tor}_p(N, M) & q = 0 \end{cases}$$

The answer as to how these two ways of computing the homology comes to us through spectral sequences.

Definition 7.8 (fake)**.** A spectral sequence is a sequence of pages $E_r^{p,q}$ along with differential $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$ such that $E_{r+1} \cong \ker(d_r)/\mathrm{im}(d_r)$ **squares to zero**

Under good conditions, this stabilizes to an E_∞ page, which hopefully contains information about some graded object H_* that we want to understand.

What this means that there is some exhaustive filtration on H_*, denoted F_*H_* such that $E_\infty^{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$.

(filtration by $p + q = k$, $E_{0,k}$ to $E_{k,0}$)

That is, we can only determine what we want up to extensions.

Always subquotient of E_0 page, in our case it is first quadrant, and note spacing of differentials.

Theorem 7.9. *Given a double chain complex D, there is a homological spectral sequence with*

$$E^2 \cong H_H(H_V(D))$$

which converges under mild conditions to the

$$H_*(Tot(D))$$

In particular, if this spectral sequence is afirst quadrant, then it converges.

Example 7.10 Let M be a module over a ring R. Let C be a chain complex of free R-modules, and let $M \to I$ be an injective resolution.

Then we have a first quadrant spectral sequence coming from the double complex $\mathrm{Hom}(C_p, I_q)$, with

$$E^2 \cong \mathrm{Ext}^q(H_p(C), M)$$

converging to

$$H^*(C; G)$$

You can recover the UCT for a PID case.

Example 7.11 Tor is symmetric.

Example 7.12 Prove the Snek lemma (arrows going up, horizontal tells us 0. figure out E_1, then know E_2 looks like, and what the maps must be so tells us exactness and iso of coker and ker

8. Thursday Exercises

Exercise 8.1. Formulate the Künneth spectral sequence, generalizing the Künneth theorem that we saw on Tuesday:

Theorem 8.2 (Künneth)**.** *Let R be a principal ideal domain, Let C be a chain complex of flat (read:free) R-modules, and D be any chain complex. We would like to compute the homology groups $H_k(C \otimes_R D)$.*

For each k, there is a natural short exact sequence of abelian groups

$$0 \to \bigoplus_{i+j=k} H_i(C) \otimes_R H_j(D) \to H_k(C \otimes_R D) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0$$

(Hint: it arises as a certain double complex).

Exercise 8.3. Use the spectral sequence of a double complex to prove the five lemma.

Exercise 8.4. Use the spectral sequence of a double complex to show that a short exact sequence of complexes induces a long exact sequence in cohomology

Exercise 8.5. Use the spectral sequence of a double complex to show that given a map $f : X \to Y$ of complexes, then the cone of f, $C(f)$, fits into a long exact sequence in homology.

Exercise 8.6. Show that if D is a first quadrant double complex, and if either all rows or all columns are exact, then so is $\mathrm{Tot}(D)$.

Exercise 8.7. Figure out how a cohomological spectral sequence of double cochain complexes should look like, and use it to prove the other forms of the Universal Coefficient Theorems seen in the Tuesday exercises.

9. Friday

Today we'll talk about the Leray-Lyndon-Hochschild-Serre Spectral Sequence. This is a spectral sequence that tells us homological/cohomological information about a fibration of topological space.

Definition 9.1. A map of (CW complexes) topological spaces $E \xrightarrow{f} B$ is called a Serre fibration if it satisfies homotopy lifting for all cell complexes K:

$$\begin{array}{ccc} K & \xrightarrow{g} & E \\ \downarrow{\scriptstyle i} & \nearrow^{l} & \downarrow{\scriptstyle f} \\ K \times I & \xrightarrow[1\mapsto n]{} & B \end{array}$$

We denote the fiber of f by $F := \pi^{-1}(b)$, and write $F \to E \to B$.

Example 9.2 Fibrations generalize the notion of a fiber bundle: except the fibers are only homotopy equivalent (no local cartesian, but allows movement across fibers).

Example 9.3 Path space fibration $\Omega S^n \to \mathrm{Map}_*(I, X) \to X$.

Example 9.4 G, H Lie groups, H closed, fibration $H \to G \to G/H$.

Proposition 9.5. Any map $f : X \to Y$ can be decomposed into the composition of homotopy equivalence followed by a Serre fibration. Middle space is pullback of $Y^I \to Y \xleftarrow{f} X$.

Proposition 9.6. $\pi(B)$ acts on F.

like deck transformation, loops in X lift to a path representing it in E. acts on E, and acts on F by permuting it.

this induces an action on singular n-chains on E by post composition. This makes $C_n(E)$ and $C_n(F)$ a $\mathbb{Z}\pi(X)$ module, so we need to define co/homology groups with local coefficients $H_n(E; M)$, $H^n(E; M)$ in the usual way by taking hom/tensor with a $\mathbb{Z}\pi(X)$ module M but with the chains of the universal cover.

Proposition 9.7. If M has a trivial $\mathbb{Z}\pi(X)$ structure, then $H_n(X; M)$ is just ordinary homology with coefficients in M. for a simplex, lifts (of $C_n(X)$ to $C_n(\widetilde{(X)})$ form an orbit of π, and these are identified (can multiply by element and get same thing).

The spectral sequence will mostly go through if we work with simple fibrations of (cubical chains) CW complexes $F \to E \to B$, where $\pi(B)$ acts trivially.

Proposition 9.8. The reason we need a Serre fibration is because a fibration induces a long exact sequence of homotopy groups

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \cdots \to \pi_0(E)$$

Theorem 9.9 (Homological Serre Spectral Sequence)**.** *Given a Serre fibration $F \to E \to B$ and a $\mathbb{Z}[\pi(B)]$-module M, there is a spectral sequence*

$$E^2_{p,q} = H_q(B; H_q(F; M)) \Rightarrow H_{p+q}(E; M)$$

with differential $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

filtration on $C_*(E)$ by $F_pC_*(E)$ singular chains supported in $\pi_{-1(B_p)}$. So associated graded is $C_*(\pi_{-1(B_p)},\pi_{-1(B_{p-1})})$ but skeleton quotient is wedge over S^p, so wedge of copies of $S^p \wedge F_+$. So $H_q(F)$ over number of cells, so $C_p(B) \otimes H_q(F)$ isomorphic as chain complexes to E_1 term

Theorem 9.10 (Cohomological Serre Spectral Sequence)**.** *Given a Serre fibration $F \to E \to B$ and a $\mathbb{Z}[\pi(B)]$-module M, there is a spectral sequence*

$$E_2^{p,q} = H^q(B; H^q(F; M)) \Rightarrow H^{p+q}(E; M)$$

with differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$

this has more structure since H^* is a graded ring (it is a SS of DGAs).

Example 9.11 S^1 is $K(\mathbb{Z}, 1)$, compute $H_*(K(\mathbb{Z}, 2)$. Path space fibration, rows 0,1, only lower left survives, so d_2 must be iso. so $\mathbb{Z}$ in even degrees, 0 for odd

$K(\mathbb{Z}, 2)$ is $\mathbb{C}P^\infty$.

Example 9.12 Homology of ΩS^n from path space fibration.

Example 9.13 Consider fibration $U(n-1) \to U(n) \to S^{2n-1}$. $U(n)$ (conjugate transpose is inverse : g in gln(C) that preserve inner product) acts transitively on $S^{2n-1} \in \mathbb{C}^n$. Pick stabilizer of vector for fiber.

We know $U(1) = S^1$, so induct on n. differentials are zero for degree reasons

10. Friday Exercises

Exercise 10.1. Show that the cohomology of X with local coefficients in $\mathbb{Z}[\pi(X)]$ is isomorphic to the cohomology of the universal cover of X, $\tilde{X}$. That is,

$$H_n(X; \mathbb{Z}[\pi(X)]) \cong H_n(\tilde{X})$$

Exercise 10.2. Show that $H^n(G; k) \cong H^n(BG; k)$.

Exercise 10.3. Let $1 \to N \to G \to G/N \to 1$ be a SES of groups, and let M be a G-module (M is an abelian group with a G-action that distributes over addition).

Show that we have the Lyndon-Hochschild-Serre spectral sequence, with

$$E^2_{p,q} = H^q(G/N; H^q(N; M)) \Rightarrow H^{p+q}(G; M)$$

Exercise 10.4. Show that a (Serre) fibration $F \to E \to B$ induces a long exact sequence of homotopy groups

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \cdots \to \pi_0(E)$$

Exercise 10.5. Prove(recover) the Hurewicz isomorphism using the path fibration.

Exercise 10.6. Prove(recover) the Eilenberg-Zilber Theorem.

Exercise 10.7. Play around with the fibration $S^n \to D^n \to S^{n+1}$.

Exercise 10.8. Play around with the Hopf fibration $S^1 \to S^3 \to S^2$.

Exercise 10.9. Compute the homology and cohomology of the degree p map from $S^n \to S^n$. (assuming we have replaced it with a Serre fibration).

Exercise 10.10.

Definition 10.11. Let $V_2(\mathbb{R}^{n+1})$ be the space of orthogonal pairs of vectors in $\mathbb{R}^{n+1}$.

(1) Show we have a Serre fibration $S^2 \to V_2(\mathbb{R}^{n+1}) \to S^n$
(2) Compute $H^*(V_2(\mathbb{R}^{n+1}))$.

Department of Mathematics, University of Texas at Austin, Austin, TX 78751
Current address: Department of Mathematics, University of Texas at Austin, Austin, TX 78751
E-mail address: richard.wong@math.utexas.edu