Spectral Sequence Training Montage, Day 4

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Summer Minicourses 2020

Slides, exercises, and video recordings can be found at
https://web.ma.utexas.edu/SMC/2020/Resources.html

Let G be a finite group, and X a spectrum with G-action. Recall that we can form the homotopy fixed point spectum

$$X^{hG} = F((EG)_+, X)^G$$

Theorem

We have the homotopy fixed point spectral sequence, which takes in input the spectrum X with a G-action, and computes $\pi_*(X^{hG})$:

$$E_2^{s,t}(R) = H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG})$$

Example

Let G be a finite group and M a G-module. This induces a G-action on the Eilenberg-Maclane spectrum HM. Then we have

$$\pi_{-*}(HM^{hG}) \cong H^*(G;M)$$

Example

For $p = 2$, $\pi_*((H\mathbb{F}_2)^{h(\mathbb{Z}/2)^n}) \cong \mathbb{F}_2[x_1, \ldots, x_n]$ with $|x_i| = -1$.

For p odd, $\pi_*((H\mathbb{F}_p)^{h(\mathbb{Z}/p)^n}) \cong \mathbb{F}_p[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n)$ with $|x_i| = -2, |y_i| = -1$.

Let G be a finite group and M a G-module. Dual to the notion of group cohomology, there is a notion of group homology.

$$H_n(G; M) := H_n(BG; M)$$

Just as $H^0(G; M) = M^G$ (the G-fixed points), we have that $H_0(G; M) = M_G$ (the G-orbits).

How is group homology $H_*(G; M)$ related to group cohomology $H^*(G; M)$?

Recall that there is the **norm map**:

$$N_G : M_G \to M^G$$

$$[x] \mapsto \sum_{g \in G} g \cdot x$$

If we think of $H_*(G; M)$ and $H^*(G; M)$ in terms of $\mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M)$ and $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M)$, we can stitch together group homology and cohomology via the norm map to form **Tate cohomology**,

$$\widehat{H}^i(G; M) \cong \begin{cases} H^i(G; M) & i \geq 1 \\ \mathrm{coker}(N_G) & i = 0 \\ \ker(N_G) & i = -1 \\ H_{-i-1}(G; M) & i \leq -2 \end{cases}$$

There is also a notion of **homotopy orbits**

$$k_{hG} = (EG_+ \wedge X)^G$$

Theorem

We have the homotopy orbit spectral sequence, which takes in input the spectrum X with a G-action, and computes $\pi_*(X_{hG})$:

$$E_2^{s,t}(R) = H_s(G; \pi_t(X)) \Rightarrow \pi_{s+t}(X^{hG})$$

Proposition

Let G be a finite group and M a G-module. There is an isomorphism

$$\pi_*((HM)_{hG}) \cong H_*(G; M)$$

Just like there is a norm map in group cohomology, there is a norm map $N_G : X_{hG} \to X^{hG}$.

We build it out of the cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$. Smashing with $F(EG_+, X)$, we get

$$EG_+ \wedge F(EG_+, X) \to F(EG_+, X) \to \widetilde{EG} \wedge F(EG_+, X)$$

Taking G fixed points, we obtain a cofiber sequence

$$X_{hG} \to X^{hG} \to (\widetilde{EG} \wedge F(EG_+, X))^G$$

Definition

The **Tate fixed points** are the cofiber of the norm map:

$$X_{hG} \xrightarrow{N_G} X^{hG} \to X^{tG}$$

Theorem

We have the Tate fixed point spectral sequence, which takes in input the spectrum X with a G-action, and computes $\pi_*(R^{tG})$:

$$E_2^{s,t}(X) = \widehat{H}^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{tG})$$

Proposition

We have a multiplicative map of spectral sequences

$$\begin{array}{ccc} H^s(G, \pi_t(X)) & \longrightarrow & \widehat{H}^s(G, \pi_t(X)) \\ \Downarrow & & \Downarrow \\ \pi_{t-s}(X^{hG}) & \longrightarrow & \pi_{t-s}(X^{tG}) \end{array}$$

Remark

This map of spectral sequences is an isomorphism on $E_2^{s,t}$ *for* $s \geq 1$*, and there is an exact sequence*

$$0 \to \widehat{H}^{-1}(G;\pi_t(X)) \to H_0(G;\pi_t(X)) \xrightarrow{N_G} H^0(G;\pi_t(X)) \to \widehat{H}^0(G;\pi_t(X)) \to 0$$

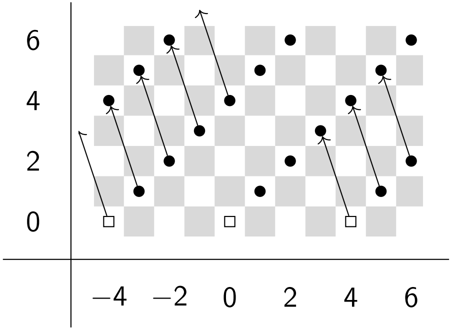
The multiplication of elements in negative degrees in $\pi_*(k^{tG})$ *is the same as the multiplication in* $\pi_*(k^{hG})$*.*

We can also compare the Tate SS to the HOSS, but multiplication in positive degrees is more complicated. For example, if G *is an elementary abelian group of* p*-rank* ≥ 2*,*

$$\pi_n(k^{tG}) \cdot \pi_m(k^{tG}) = 0$$

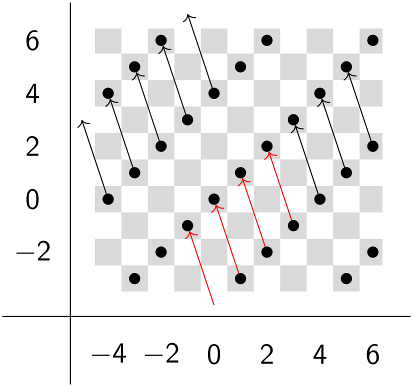
for all $n, m > 0$*.*

$$E_2^{s,t} = H^s(\mathbb{Z}/2; \pi_t(KU)) \Rightarrow \pi_{t-s}(KU^{h\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-HFPSS computing $\pi_*(KU^{h\mathbb{Z}/2}) \cong \pi_*(KO)$.
$\square = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.

$$E_2^{s,t} = \widehat{H}^s(\mathbb{Z}/2; \pi_t(KU)) \Rightarrow \pi_{t-s}(KU^{t\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-Tate SS computing $\pi_*(KU^{t\mathbb{Z}/2})$. $\bullet = \mathbb{Z}/2$.

Definition

A map $f : R \to S$ of E_∞-ring spectra is a G-**Galois extension** if the maps

(i) $i : R \to S^{hG}$

(ii) $h : S \otimes_R S \to F(G_+, S)$

are weak equivalences.

Definition

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is said to be **faithful** if the following property holds:

If M is an R-module such that $S \otimes_R M$ is contractible, then M is contractible.

Example

$KO \to KU$ is a $\mathbb{Z}/2$-Galois extension of ring spectra.

Proposition (Rognes)

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is faithful if and only if the Tate construction S^{tG} is contractible.

Example

$KO \to KU$ is a **faithful** $\mathbb{Z}/2$-Galois extension of ring spectra.

Let $R \to S$ be a faithful G-Galois extension of E_∞-rings.

Corollary

We have the homotopy fixed point spectral sequence, which takes in input the spectrum pic(S) and has E_2 page:

$$H^s(G; \pi_t(pic(S))) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ is $\mathrm{Pic}(R)$.

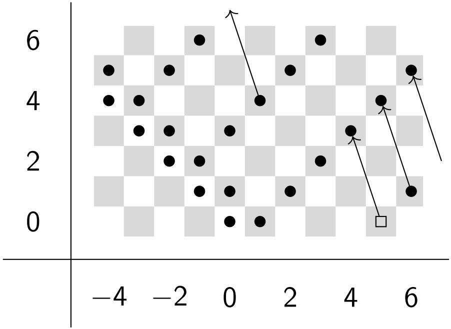
Theorem (Mathew-Stojanoska)

If $t - s > 0$ and $s > 0$ we have an equality of HFPSS differentials

$$d_r^{s,t}(\mathfrak{pic} S) \cong d_r^{s,t-1}(S)$$

Furthermore, this equality also holds whenever $2 \le r \le t - 1$.

$$E_2^{s,t} = H^s(\mathbb{Z}/2; \pi_t(\mathfrak{pic}(KU))) \Rightarrow \pi_{t-s}((\mathfrak{pic}(KU))^{h\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-HFPSS computing $\pi_*((\mathfrak{pic}(KU))^{h\mathbb{Z}/2})$. $\square = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$. Not all differentials are drawn.

Problem Session

You can find the exercises at https://web.ma.utexas.edu/SMC/2020/Resources.html.

We are using the free (sign-up required) A Web Whiteboard website. The link will be posted in the chat, as well as on the slack channel.

Future problem sessions will be from 1-1:30pm and 2:30-3pm CDT.