SPECTRAL SEQUENCES TRAINING MONTAGE EXERCISES

ARUN DEBRAY AND RICHARD WONG

Abstract. These are exercises designed to accompany the 2020 Summer Minicourse "Spectral Sequence Training Montage", led by Arun Debray and Richard Wong. Spectral sequences covered included the Serre SS, the homotopy fixed-point SS, the Atiyah-Hirzebruch SS, the Tate SS, and the Adams SS. Minicourse materials can be found at https://web.ma.utexas.edu/SMC/2020/Resources.html.

Instructor's note: I compiled this list of exercises because there is simply too much material to cover in a one week minicourse. The topics covered in these exercises include: background material; interesting calculations; interesting applications; and questions for your own enlightenment. For each day, I will recommend a subset of exercises that I think are the most important.

1. Monday Exercises

The section on fibrations is background material. I recommend exercises 1.5, 1.6., 1.8, 1.10, 1.13, and 1.17.

1.1. Fibration exercises

Exercise 1.1. Show that if $f : X \to B$ is a Serre fibration with B path-connected, then the fibers over any two points are homotopy equivalent. That is, $f^{-1}(b_1) \simeq f^{-1}(b_2)$.

Exercise 1.2. Show that a Serre fibration $F \to E \to B$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E)$$

Exercise 1.3. Given a short exact sequence of groups $H \to G \to G/H$, show that there is a Serre fibration of classifying spaces $BH \to BG \to BG/H$

Exercise 1.4. Show that the notion of Serre fibration is strictly weaker than the notion of a Hurewicz fibration.

Exercise 1.5. Show that the fibration $G \to EG \to BG$ can be obtained from the path space fibration $\Omega BG \to BG^I \to BG$.

Exercise 1.6. Given a universal cover $\tilde{X} \to X$ with $\pi_1(X) = G$, show that we have a fibration $\tilde{X} \to X \to BG$.

In general, If G acts on a space X such that the quotient map $X \to X/G$ is a covering space, show that we have a fibration $X \to X/G \to BG$.

1.2. Spectral Sequence Computations

Exercise 1.7. Given a universal cover $\tilde{X} \to X$ with $\pi_1(X) = G$, use the Serre spectral sequence to show that there is an isomorphism $H^*(X; \mathbb{Q}) \to (H^*(\tilde{X}; \mathbb{Q}))^G$.

How can this statement be generalized? For example, how necessary is the coefficient ring $\mathbb{Q}$?

Exercise 1.8. Show that if $F \to E \to B$ is a Serre fibration with $\pi_1(B)$ acting trivially, and we take coefficients $A = k$ for some field k, then the Serre spectral sequence takes the form

$$E_2^{s,t} = H^p(B; k) \otimes H^q(F; k) \Rightarrow H^{p+q}(E; k)$$

Exercise 1.9. Play around with the Serre spectral sequence for the Hopf fibration $S^1 \to S^3 \to S^2$.

Exercise 1.10. Play around with the Serre spectral sequence for the fibration $U(n-1) \to U(n) \to S^{2n-1}$.

Exercise 1.11. Play around with the Serre spectral sequence for the fibration $SO(n) \to SO(n+1) \to S^n$.

Exercise 1.12. Let $V_2(\mathbb{R}^{n+1})$ be the space of orthogonal pairs of vectors in $\mathbb{R}^{n+1}$.

(1) Show we have a Serre fibration $S^2 \to V_2(\mathbb{R}^{n+1}) \to S^n$
(2) Compute $H^*(V_2(\mathbb{R}^{n+1}))$.

Exercise 1.13. Compute the cup product structure on $H^*(\Omega S^n)$ using the path space fibration $\Omega S^n \to (S^n)^I \to S^n$.

Exercise 1.14. Compare the spectral sequence for the fibration $S^2 \to S^2 \times S^2 \to S^2$ with the fibration $S^2 \to X \to S^2$, where X is built by taking two mapping cylinders of the Hopf map $S^3 \to S^2$, and gluing them together along the identity on S^3.

Show that $H^*(S^2 \times S^2)$ and $H^*(X)$ have different ring structures.

Exercise 1.15. Prove (recover) the Gysin sequence.

Theorem (The Gysin Sequence)**.** Let $S^n \to E \to B$ be a Serre fibration with B simply connected and $n \geq 1$. There exists a long exact sequence

$$\cdots \to H^k(B) \to H^k(X) \to H^{k-n}(B) \to H^{k+1}(B) \to \cdots$$

Exercise 1.16. Prove (recover) the Wang sequence.

Theorem (The Wang Sequence)**.** Let $F \to X \to S^n$ be a Serre fibration with B simply connected and $n \geq 1$. There exists a long exact sequence

$$\cdots \to H^{k-1}(F) \to H^{k-n}(F) \to H^k(X) \to H^k(F) \to \cdots$$

Exercise 1.17. Prove (recover) this Hurewicz isomorphism using the path fibration

$$\Omega(X) \to PX \to X$$

Theorem (Hurewicz)**.** Let X be an $(n-1)$-connected space, with $n \geq 2$. Then $\tilde{H}_i(X) = 0$ for $i \leq n-1$, and we have the Hurewicz isomorphism

$$\pi_n(X) \cong H_n(X)$$

Exercise 1.18. Prove (recover) the Leray-Hirsch Theorem.

Theorem (Leray-Hirsch)**.** Let $F \to E \to B$ be a fiber bundle such that F is of finite type. That is, that $H^p(F; \mathbb{Q})$ is finite dimensional for all p.

Furthermore, assume that the inclusion $i : F \to E$ induces a surjection

$$i^* : H^*(E; \mathbb{Q}) \to H^*(F; \mathbb{Q})$$

Then we have an isomorphism of $H^*(B; \mathbb{Q})$-modules

$$H^*(F; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(B; \mathbb{Q}) \cong H^*(E; \mathbb{Q})$$

Exercise 1.19. How can the Leray-Hirsch theorem above be generalized? In particular, how necessary is the coefficient ring $\mathbb{Q}$?

1.3. **For your enlightenment**

Exercise 1.20. Show that there is a relationship between the bigraded chain complex

$$\cdots \to H^*(E_{s-1}, E_s) \xrightarrow{d} H^*(E_s, E_{s-1}) \xrightarrow{d} H^*(E_{s+1}, E_s) \to \cdots$$

and $H^*(B)$ and $H^*(F)$.

Namely, that there is an isomorphism

$$E_1^{s,t} \cong C^s(B; H^t(F))$$

where $C^*(B; H^t(F))$ is the cellular cochain complex for B with coefficients in $H^t(F)$.

Exercise 1.21. What was special about the Serre filtration on X? Can you construct exact couples using a different filtration? Can you construct a spectral sequence using a different filtration?

Exercise 1.22. What was special about using cohomology? Can you construct a homological Serre spectral sequence?

Can you construct a spectral sequence using a generalized cohomology theory?

Department of Mathematics, University of Texas at Austin, Austin, TX 78751
Current address: Department of Mathematics, University of Texas at Austin, Austin, TX 78751
E-mail address: richard.wong@math.utexas.edu