SPECTRAL SEQUENCES TRAINING MONTAGE EXERCISES

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Abstract. These are exercises designed to accompany the 2020 Summer Minicourse "Spectral Sequence Training Montage", led by Arun Debray and Richard Wong. Spectral sequences covered included the Serre SS, the homotopy fixed-point SS, the Atiyah-Hirzebruch SS, the Tate SS, and the Adams SS. Minicourse materials can be found at https://web.ma.utexas.edu/SMC/2020/Resources.html.

Instructor's note: I compiled this list of exercises because there is simply too much material to cover in a one week minicourse. The topics covered in these exercises include: background material; interesting calculations; interesting applications; and questions for your own enlightenment. For each day, I will recommend a subset of exercises that I think are the most important.

1.

2. Tuesday Exercises

There is subsection on groups acting freely on spheres, which is an interesting application of group cohomology. I recommend exercises 2.1, 2.2, 2.3, 2.10, 2.13, and 2.14.

2.1. Group Cohomology

Exercise 2.1. Show that there is an isomorphism

$$H^*(G; M) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us an algebraic way to compute group cohomology.

Exercise 2.2. Let M be a $\mathbb{Z}G$-module. Show that $H^0(G; M) = M^G$, the G-fixed points of M.

Exercise 2.3. Compute the LHS spectral sequence with coefficients in a field of characteristic p for the fibration $B(\mathbb{Z}/2)^2 \to BA_4 \to B\mathbb{Z}/3$

Exercise 2.4. Compute $H^*(BD_8; \mathbb{F}_2)$ using the LHS spectral sequence.

Exercise 2.5. Compute $H^*(BQ_8; \mathbb{F}_2)$ using the LHS spectral sequence.

Exercise 2.6. Show that for G a finite group, and a faithful unitary representation $\Phi : G \to U(n)$ with Chern classes $c_i(\Phi)$, then

$$|G| \Big| \prod_{i=1}^{n} exp(c_i(\Phi))$$

Hint: Consider the fibration $G \to U(n) \to U(n)/G$.

Exercise 2.7. Let G be a finite group of order n. Show that $n \cdot H^i(G; M) = 0$ for any G-module M. That is, that group cohomology is always $|G|$-torsion.

Hint: consider the restriction and transfer maps $\mathrm{res}^G_H : H^*(G; M) \to H^*(H; M)$ and $\mathrm{tr}^G_H : H^*(H; M) \to H^*(G; M)$. Show that the composite $\mathrm{tr}^G_H \circ \mathrm{res}^G_H$ is multiplication by the index $|G : H|$.

2.2. Groups acting on Spheres

Exercise 2.8. Show that $\mathbb{Z}/n$ are the only finite groups that act freely on S^1.

Exercise 2.9. Show that if n is even, then the only non-trivial finite group that can act freely on S^n is $\mathbb{Z}/2$.

Exercise 2.10. A finite group G is **periodic** of period $k > 0$ if $H^i(G; \mathbb{Z}) \cong H^{i+k}(G; \mathbb{Z})$ for all $i \geq 1$, where $\mathbb{Z}$ has trivial G action.

Show that if G acts freely on S^n, then G is periodic of period $n + 1$.

Exercise 2.11. Show that $\mathbb{Z}/p \times \mathbb{Z}/p$ does not act freely on S^n:

Exercise 2.12. Show that not every periodic group with period 4 acts freely on S^3. (Consider $G = S_3$).

2.3. The HFPSS

Exercise 2.13. Consider the fiber sequence of spaces

$$G/N \to BN \to BG$$

, and the morphism of ring spectra $k^{hG} \to k^{hN}$ obtained by taking cochains with Hk-valued coefficients.

Compare this HFPSS with the Serre spectal sequence.

Exercise 2.14. Consider the fiber sequence $S^1 \to B\mathbb{Z}/2 \to BS^1$. Taking cochains with $\mathbb{F}_2$-valued coefficients, we obtain a morphism of ring spectra $H\mathbb{F}_2^{hS^1} \to H\mathbb{F}_2^{h\mathbb{Z}/2}$.

Compute the HFPSS for the S^1-action on $H\mathbb{F}_2^{h\mathbb{Z}/2}$

Exercise 2.15. Let p be an odd prime. Consider the fiber sequence $S^1 \to B\mathbb{Z}/p \to BS^1$. Taking cochains with $\mathbb{F}_p$-valued coefficients, we obtain a morphism of ring spectra $H\mathbb{F}_p^{hS^1} \to H\mathbb{F}_p^{h\mathbb{Z}/p}$.

Compute the HFPSS for the S^1-action on $H\mathbb{F}_p^{h\mathbb{Z}/p}$

2.4. For your enlightenment

Exercise 2.16. Construct BG for a topological group G. Is $BG \simeq K(G,1)$?

Exercise 2.17. If G is a Lie group, what conditions on a subgroup H give a fibration $BH \to BG \to BG/H$?

Exercise 2.18. Generalize the idea of cohomology with local coefficients for a space X with universal cover $\tilde{X}$.

Exercise 2.19. Show that the cohomology of X with local coefficients in $\mathbb{Z}[\pi(X)]$ is isomorphic to the cohomology of the universal cover of X, $\tilde{X}$. That is,

$$H_n(X; \mathbb{Z}[\pi(X)]) \cong H_n(\tilde{X})$$

Exercise 2.20. Dual to the notion of group cohomology, there is a notion of group homology.

Show that there is an isomorphism

$$H_*(G; M) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us an algebraic way to compute group homology.

Exercise 2.21. Let M be a $\mathbb{Z}G$-module. Show that $H_0(G; M) = M_G$, the G-orbits or coinvariants of M. In other words, M_G is the quotient of M by the submodule generated by elements of the form $g \cdot m - m$.

Exercise 2.22. Dual to the notion of homotopy fixed points, there is a notion of homotopy orbits.

Construct the homotopy orbit spectral sequence.

Exercise 2.23. Show that for R a ring, and G acting trivially on the Eilenberg-Maclane spectrum HR, there is an isomorphism

$$\pi_*((HR)_{hG}) \cong H_*(G; R)$$

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