SPECTRAL SEQUENCES TRAINING MONTAGE EXERCISES

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Abstract. These are exercises designed to accompany the 2020 Summer Minicourse "Spectral Sequence Training Montage", led by Arun Debray and Richard Wong. Spectral sequences covered included the Serre SS, the homotopy fixed-point SS, the Atiyah-Hirzebruch SS, the Tate SS, and the Adams SS. Minicourse materials can be found at https://web.ma.utexas.edu/SMC/2020/Resources.html.

Instructor's note: I compiled this list of exercises because there is simply too much material to cover in a one week minicourse. The topics covered in these exercises include: background material; interesting calculations; interesting applications; and questions for your own enlightenment. For each day, I will recommend a subset of exercises that I think are the most important.

1.

2. Tuesday Exercises

3.

4. Thursday Exercises

4.1. Trickier Serre spectral sequence questions

Exercise 4.1. Similarly to the example given in lecture today, investigate $H^*(G;\mathbb{Z}_{w_1(\rho)})$ in the first few degrees using the multiplicative structure when combined with $H^*(G;\mathbb{Z})$, for the following groups.

(1) $G = \mathrm{O}(2)$, and ρ is the standard two-dimensional real representation. The extension is $1 \to \mathrm{SO}(2) \to \mathrm{O}(2) \to \mathbb{Z}/2 \to 1$.
(2) $G = D_{2n}$, and ρ is the two-dimensional real representation of rotations and reflections. The extension is $1 \to C_n \to D_{2n} \to \mathbb{Z}/2 \to 1$. Notes:
   - These spectral sequences are compatible for different n, and via $D_{2n} \to \mathrm{O}(2)$, with the spectral sequence in the previous part of the problem.
   - The spectral sequences will depend on the parity of n, and probably also on n mod 4.
   - The integral cohomology ring of D_{2n} is given here: `https://math.stackexchange.com/questions/1294806`.

Exercise 4.2. Can you prove that $H^*(\mathbb{Z}/2;R) \cong \mathbb{Z}[e]/(2e)$ with $|e| = (1,-)$?

Exercise 4.3. Challenge question: let's compute the Stiefel-Whitney classes of the *Wu manifold* $W := \mathrm{SU}(3)/\mathrm{SO}(3)$. This will prove that W is not null-bordant; in fact, it is the generator of $\Omega_5^{\mathrm{O}} \cong \mathbb{Z}/2$, representing the lowest-degree element that can't be built using real projective spaces.

(1) This part isn't as hard: use the Serre spectral sequence to show that $H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2,z_3]/(z_2^2,z_3^2)$, with $|z_i| = i$. Notes: there is a diffeomorphism $\mathrm{SO}(3) \cong \mathbb{RP}^3$, and $H^*(\mathrm{SU}(3);\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3,x_5]/(x_3^2,x_5^2)$ with $|x_i| = i$. (Can you prove this with another Serre spectral sequence argument?)
(2) Now, use the method of Borel-Hirzebruch, as outlined in `https://math.stackexchange.com/questions/581401`, to compute the Stiefel-Whitney classes of W.

4.2. Tate Cohomology / SS Exercises

Exercise 4.4. Let $G \cong \mathbb{Z}/p$. Compute group homology $H_*(G;\mathbb{F}_p)$ for the trivial G action.

Exercise 4.5. Let $G \cong \mathbb{Z}/p$. Compute the norm map for the trivial G action on $\mathbb{F}_p$.

Exercise 4.6. Let $G \cong \mathbb{Z}/p$. Compute the Tate cohomology $\widehat{H}^*(G;\mathbb{F}_p)$ for the trivial G action.

Exercise 4.7. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute group homology $H_*(G;\mathbb{Z})$ for the trivial G action.

Exercise 4.8. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute the norm map for the trivial G action on $\mathbb{Z}$.

Exercise 4.9. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute the Tate cohomology $\widehat{H}^*(G;\mathbb{Z})$ for the trivial G action.

Show that for all $n \in \mathbb{Z}$, there is an isomorphism

$$\widehat{H}^n(G;\mathbb{Z}) \cong \widehat{H}^{n+2}(G;\mathbb{Z})$$

Exercise 4.10. Let G be a finite group such that $p \mid |G|$. Show that the map of ring spectra induced by the fiber sequence $G \to EG \to BG$

$$(H\mathbb{F}_p)^{hG} \to H\mathbb{F}_p$$

is a **non-faithful** Galois extension of ring spectra.

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