The Adams spectral sequence over $\mathscr{A}(1)$

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Overview

- A few general things about the Adams spectral sequence
- Working over $\mathscr{A}(1)$ to compute *ko*-theory
- Computing the E_2-page
- Differentials and extensions

Why we're doing what we're doing today

- The Adams spectral sequence is difficult (e.g. Mahowald's uncertainty principle)
- There's no way to cover an introduction in an hour suitable for working on computations
- In some cases, the Adams spectral sequence dramatically simplifies, but still carries much of the same structure
  - This is a useful pedagogical example
  - …but it's not just a toy: calculations over $\mathscr{A}(1)$ do appear in papers

The general Adams spectral sequence

- Pick two spectra X and Y, and a prime p, and let $\mathscr{A}_p$ be the mod p Steenrod algebra
- The Adams spectral sequence has signature

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \Longrightarrow [Y,X]^\wedge_p.$$

- That is, it computes stable homotopy classes of maps
- Often, $Y = \mathbb{S}$, so it computes stable homotopy groups.
  $Y = X = \mathbb{S}$ is the most commonly studied example, by far
  - For $Y = X = \mathbb{S}$, the Adams spectral sequence is understood up to about $t - s = 63$, and blearily up to about degree $t - s = 90$ (Adams, May, Mahowald, Barratt, Bruner, Tangora, Isaksen-Wang-Xu, ...)

Variants on the Adams spectral sequence

- You could also begin with some other spectrum E and (assuming something I'll skate over) obtain an "E-based Adams spectral sequence" with E_2-page $\mathrm{Ext}^{s,t}_{E^*E}(E^*(X), E^*(Y))$
- We took $E = H\mathbb{F}_p$, and $E = ko$ is sometimes considered
- $E = BP$ is called the *Adams-Novikov spectral sequence* and is fairly commonly studied

Simplifying the Adams spectral sequence

- $\mathscr{A}$ is noncommutative and not finitely generated, a.k.a. not fun
- We will work with a simpler replacement for which the key features of the Adams spectral sequence still apply
- Let $\mathscr{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \subset \mathscr{A}_2$ — 8-dimensional, still noncommutative

Change-of-rings

- Stong calculated that

$$\widetilde{H}^*(ko;\mathbb{F}_2) \cong \mathscr{A} \otimes_{\mathscr{A}(1)} \mathbb{F}_2$$

- There is a canonical isomorphism

$$\mathrm{Hom}_A(B \otimes_C D, E) = \mathrm{Hom}_C(D, E)$$

and the derived version is

$$\mathrm{Ext}_A^{s,t}(B \otimes_C D, E) = \mathrm{Ext}_C^{s,t}(D, E)$$

- In conclusion,

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*(ko\wedge X;\mathbb{F}_2),\mathbb{F}_2) = \mathrm{Ext}^{s,t}_{\mathscr{A}(1)}(\widetilde{H}^*(X;\mathbb{F}_2);\mathbb{F}_2)$$

- That is, if you want to compute the 2-completed *ko*-theory of X, you can run the Adams spectral sequence over $\mathscr{A}(1)$, which is much simpler!
- The 7-connected map $MSpin \to ko$ means this also applies to spin bordism and variants
- We'll work with this simplification

The structure of Ext

- $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$ is an algebra: elements of $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$ are represented by extensions $\Sigma^t \mathbb{F}_2 \to P_s \to \cdots \to P_1 \to \mathbb{F}_2$. The multiplication is the *Yoneda product* compose two extensions
- In the same way, $\mathrm{Ext}(M, \mathbb{F}_2)$ is a module over $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$
- Differentials commute with this algebra action, so it's useful to be aware of
- $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$ converges to $\pi_* ko$, and this module structure lifts to the $\pi_* ko$-action on $\pi_*(ko \wedge X) = \widetilde{ko}_*(X)$

$\mathrm{Ext}^{*,*}_{\mathscr{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$

So… how do you calculate Ext of stuff?

- Sometimes you can look it up (seriously!) e.g. Beaudry-Campbell
- A short exact sequence of modules induces a long exact sequence in Ext
- Sometimes you have to write down a projective resolution
- A few other tricks in very specific situations

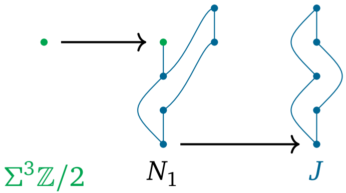
Example: the Joker

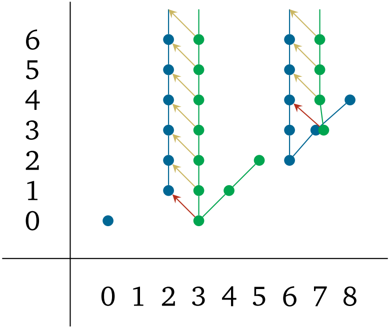
Example: Spanish question mark

Example $C\eta$

Example: long exact sequence

$$0 \longrightarrow \Sigma^3\mathbb{Z}/2 \longrightarrow N_1 \longrightarrow J \longrightarrow 0$$





Technique: use a computer program

- There are a few programs that calculate Ext over (a subalgebra of) the Steenrod algebra
- One by Bob Bruner, on his website
- Another by Hood Chatham and Dexter Chua: `https://spectralsequences.github.io/rust_webserver/`

Differentials in the Adams spectral sequence

- Adams grading: d_r moves one unit left ($t-s$ degree), r units up (s degree)
- h_0- or h_1-linearity of differentials solves a lot of problems (e.g. if $h_0x = 0$ and $h_0y \neq 0$, the $d_rx \neq y$)
- Otherwise, differentials are usually hard, even d_2s, and even over $\mathscr{A}(1)$

Margolis' theorem is your friend!

- Theorem: $\mathscr{A}(1)$ summands in $H^*(X;\mathbb{F}_2)$ correspond to $H\mathbb{F}_2$ summands of $ko \wedge X$
- Upshot: no differentials to or from Ext elements corresponding to an $\mathscr{A}(1)$ summand, and they don't participate in any nontrivial extensions
- In many cases, this (and linearity over $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$) suffices to collapse the spectral sequence on the E_2-page

Other techniques for computing differentials

- Map to or from a different spectral sequence…
- Use $\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2)$-algebra structure to propagate differentials
- In low degrees, compute with a different spectral sequence (e.g. Atiyah-Hirzebruch)

Extensions: a few tricks

- Margolis' theorem, of course
- h_0 lifts to multiplication by 2
- h_1 lifts to multiplication by $\eta \in ko_1$
- $2\eta = 0$, so if $\eta x \neq 0$, then x isn't divisible by 2. Sometimes this helps
- Otherwise extensions can be pretty tough

What changes if we're not working over $\mathscr{A}(1)$?

- Ext over your subalgebra B is still a module over $\mathrm{Ext}_B(\mathbb{F}_2, \mathbb{F}_2)$. More complicated bad, but more structure good: solves some problems
- Differentials commute with this algebra action
- The module action lifts analogously to ko_* acting on $ko_*(X)$

Working over other subalgebras of $\mathscr{A}$

- Working over $\mathscr{A}(2) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle$ gives you $\mathit{tmf} \wedge X$
- Working over $\mathscr{E}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1 \rangle$ gives you $ku \wedge X$