The Atiyah-Hirzebruch spectral sequence

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Overview

- Generalized (co)homology, but very quickly
- The Atiyah-Hirzebruch spectral sequence; first examples
- Stable cohomology operations and the first nonzero differential

Generalized cohomology theories

- Cohomology $H^*(X;A)$ is characterized by the *Eilenberg-Steenrod axioms* guaranteeing how it behaves on homotopy equivalences, disjoint unions, cofibrations, and contractible spaces
- There are many interesting functors which satisfy all of these axioms except triviality on contractible spaces
- Called *generalized cohomology theories*
- Central subject of research in algebraic topology, and has been for decades
- Dually, there are *generalized homology theories*

Bluff your way through spectra

- Generalized cohomology theories are represented by things called *spectra*, objects of a category which is a stabilization of Top under suspensions
- So a topological space X determines a spectrum $\Sigma^\infty X$, and a spectrum determines a generalized cohomology theory

Multiplicative structures

- Ordinary cohomology is naturally a ring (on spaces, not spectra)
- We want versions of this in generalized cohomology
- And it does hold in many examples! But the theory of *ring spectra* and *multiplicative generalized cohomology theories* is subtle

Examples: ordinary cohomology

- Given an abelian group A, get an *Eilenberg-Mac Lane spectrum* HA, which represents ordinary (co)homology valued in A
- Give A a ring structure and HA becomes a ring spectrum
- Okay, honestly, this is kind of a boring example

Examples: K-theory

- There is a ring spectrum KU whose corresponding cohomology theory is *complex K-theory*
- This is a 2-periodic spectrum, meaning $\Sigma^2 KU \simeq KU$
- $\pi_*(KU) \cong \mathbb{Z}[t, t^{-1}]$, with $|t| = 2$, so $\pi_{2i}(KU) = \mathbb{Z}$ and $\pi_{2i+1}(KU) = 0$
- Often $KU^*(X)$ is denoted $K^*(X)$. $K^0(X)$ computes the Grothendieck group of complex vector bundles on X (assumed cpt Hausdorff) w.r.t. direct sum

Connective K-theory

- A variant of KU called ku, where we only keep the homotopy groups in degrees 0 and above
- Called *connective (complex) K-theory*
- $\pi_*(ku) \cong \mathbb{Z}[t]$, $|t| = 2$
- Also a ring spectrum

Real K-theory

- Using real instead of complex vector bundles, we obtain an 8-periodic ring spectrum KO
- $\pi_*(KO) \cong \mathbb{Z}[\eta, x, v, v^{-1}]$ with $|\eta| = 1$, $|x| = 4$, $|v| = 8$
- So the groups go $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}\ldots$
  - You can sing this to "Twinkle twinkle little star" and it is called the *Bott song*
- The connective version (only keep π_k for $k \geq 0$) is denoted ko

Examples: bordism

- Consider the functor which assigns to (reasonable) spaces X the commutative monoid of closed n-manifolds together with a map $f\colon M \to X$, modulo the submonoid of (M, f) that "bound," i.e. there's a compact $(n+1)$-manifold W and map $g\colon W \to X$ with $\partial W = M$ and $g|_M = f$
- This is an abelian group, and in fact this is a generalized homology theory $\Omega_n^{\mathrm{O}}(X)$
- Represented by a *Thom spectrum* denoted MO
- *Many* variants given a tangential structure: MSO and Ω_*^{SO} for oriented bordism; $MSpin$ and Ω_*^{Spin} for spin bordism, MU and Ω_*^{U} for (stably almost) complex bordism
  - Note that MG is not always a ring spectrum, e.g. $MPin^{\pm}$

The Atiyah-Hirzebruch spectral sequence

- Let E be a spectrum and X be a space (or spectrum)
- The *cohomological AHSS* has signature

$$E_2^{p,q} = H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X)$$

  and if E is a ring spectrum, this has a multiplicative structure
- The *homological AHSS* has signature

$$E_{p,q}^2 = H_p(X; E_q(\mathrm{pt})) \Longrightarrow E_{p+q}(X)$$

Don’t let coefficients trip you up

- $E_q(\mathrm{pt}) = \pi_q(E)$
- but $E^q(\mathrm{pt}) = \pi_{-q}E$!
- So, e.g. if E is connective (and X is a space), the cohomological AHSS is *fourth*-quadrant

Pictures: homological and cohomological differentials

Convergence

- If E is *connective* (all negative-degree homotopy groups vanish), the AHSS is single-quadrant, hence converges
- If X has finitely many nonvanishing cohomology groups, the AHSS converges
- If neither of these is true, things can get a little cagey
  - For example, for some n, $KO_n(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^\infty$, which is not a finitely generated abelian group!

Example: $K^*(\mathbb{CP}^n)$

Quick facts

- For any spectrum E, *reduced E-theory* of a pointed space X is $E^*(X/*)$.
- The maps $* \to X \to *$ split off $E_2^{0,*} = E^*(\mathrm{pt})$ from the rest of the AHSS: no differentials, no extension questions
- The Atiyah-Hirzebruch spectral sequence is functorial in both the spectrum E and the space X: one obtains maps on all of the E_r-pages which intertwine the differentials

The first nonzero differential: motivation

- Fix a spectrum E
- By *the first nonzero differential* for $q \in \mathbb{Z}$, we mean the first $d_r \colon E_r^{*,q} \to E_r^{*+r,q-r+1}$ that can nonzero in the AHSS for some input space or spectrum
- If $\pi_k E = 0$, of course d_2s to or from $E_2^{p,-k}$ vanish so we skip those, and so on

Layer cakes and the Atiyah-Hirzebruch spectral sequence

- The AHSS can be constructed by filtering E via its *Postnikov filtration*
- Vague idea: a spectrum is a many-tiered delicious cake, with different layers (Eilenberg-Mac Lane spectra) glued together by k-invariants (frosting)

k-invariants

- Suppose E is a spectrum with only one nonzero homotopy group $\pi_n E$. Then E is (a shift of) an Eilenberg-Mac Lane spectrum: $E \simeq \Sigma^n H\pi_n(E)$
- If E has exactly two nonzero homotopy groups, it might not be a wedge of (shifts) of EM spectra
- But it *does* fit into a fiber sequence where the other two pieces are EM spectra

k-invariants

$$\begin{array}{ccc} \Sigma^n H\pi_n(E) & \longrightarrow & E \\ & & \Big\downarrow \varphi \\ & & H\pi_0(E). \end{array}$$

To get this data, take the cofiber of φ:

$$\begin{array}{ccc} \Sigma^n H\pi_n(E) & \longrightarrow & E & & \\ & & \Big\downarrow \varphi & & \\ & & H\pi_0(E) & \xrightarrow{k} & \Sigma^{n+1} H\pi_n(E). \end{array}$$

This map k is called the *k-invariant* of E

k-invariants

- For E with just two nonvanishing htpy groups, $k = 0$ iff E is a sum of (shifts of) EM spectra
- For a general spectrum, one iterates this procedure, and in particular there are k-invariants between any i and j with $\pi_k(E) = 0$ for $i < k < j$
- These k-invariants are examples of *stable cohomology operations*

Stable cohomology operations

- A *stable cohomology operation* is a natural transformation $H^m(-;A) \to H^n(-;B)$ which commutes with the suspension isomorphism
- Friendly example: the *Bockstein* β, the connecting morphism in the long exact sequence in cohomology induced from a short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$
- Over $\mathbb{Q}$, all stable cohomology operations are trivial (i.e. scalar multiplication)
- Over $\mathbb{Z}$, all stable cohomology operations are torsion: reduce mod p, do something interesting, then Bockstein back up
- Over $\mathbb{F}_p$, more exciting

The Steenrod algebra at $p = 2$

- The set of stable cohomology operations $H^*(X;\mathbb{Z}/2) \to H^{*+k}(X;\mathbb{Z}/2)$ forms a graded $\mathbb{Z}/2$-algebra under composition, denoted $\mathscr{A}$ and called the *Steenrod algebra*
- Generated by *Steenrod squares* $\mathrm{Sq}^n : H^*(X;\mathbb{Z}/2) \to H^{*+n}(X;\mathbb{Z}/2)$, $n \geq 0$, satisfying some axioms and relations
  - For $n > 1$, definition is a bit technical

Axiomatic definition of Steenrod squares

- (implicit: group homomorphism, naturality, stability)
- $\mathrm{Sq}^0 = \mathrm{id}$ and Sq^1 is the Bockstein for $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$
- If $|x| = n$, $\mathrm{Sq}^n(x) = x^2$
- If $|x| < n$, $\mathrm{Sq}^n(x) = 0$
- The *Cartan formula*:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y),$$

  or, if $\mathrm{Sq}(x) := \mathrm{Sq}^0(x) + \mathrm{Sq}^1(x) + \cdots$, then $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\mathrm{Sq}(y)$
- Theorem: these properties uniquely characterize the Steenrod squares and their action on mod 2 cohomology of spaces

Ádem relations

- $\mathscr{A}$ is not free (in fact generated by Sq^{2^n} for all n):

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

- Summary: for any space X, $H^*(X;\mathbb{Z}/2)$ is an $\mathscr{A}$-module (*not* an $\mathscr{A}$-algebra), and pullback is always an $\mathscr{A}$-module homomorphism

The first nonzero differential in the cohomological AHSS

- Fix a spectrum E with $\pi_q(E) \neq 0$, $\pi_{q+r}(E) \neq 0$, $\pi_k(E) = 0$ for $q < k < q+r$
- Theorem: the first nontrivial differential in the cohomological AHSS from $E_{r+1}^{p,-q}$ to $E_{r+1}^{p+r+1,-q-r}$ is identified with the k-invariant $H^p(-;\pi_q(E)) \to H^{p+r+1}(-;\pi_{q+r}(E))$
- In many cases of interest, this is sufficient information!
- Higher differentials determined by higher cohomology operations, which are… hard

The first nonzero differential in the homological AHSS

- The *stable homology operation* dual to a given stable cohomology operation S is the dual under the cap product pairing
- If S raises degree by r, its dual lowers degree by r
- Theorem: first nonzero differential in the homological AHSS is the dual stable cohomology operation to the corresponding k-invariant

Some k-invariants

- In the next few slides, we'll give examples of spectra and their k-invariants
- Simplest: complex K-theory: both KU and ku
- 2-periodicity and $\pi_{\mathrm{odd}}KU = 0$ means there's just the one k-invariant, which is $\beta \circ \mathrm{Sq}^2 \circ r \colon H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$
- Confusingly, this is sometimes denoted Sq_3

k-invariants for KO and ko

- 8-periodicity and zeroes in the homotopy groups means there are four k-invariants to worry about
- $k_1 : H\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2$ is $\mathrm{Sq}^2 \circ r$
- $k_2 : H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{Z}/2$ is Sq^2
- $k_3 : H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$ is $\beta \circ \mathrm{Sq}^2$
- $k_4 : H\mathbb{Z} \to \Sigma^5 H\mathbb{Z}$ is $\beta \circ \mathrm{Sq}^4$

k-invariants for bordism theories

- Unoriented bordism: Thom showed MO is a wedge sum of shifts of Eilenberg-Mac Lane spectra
  - All k-invariants are zero
  - The Atiyah-Hirzebruch spectral sequence collapses at E_2 without extension problems
- Spin bordism: There is a 7-connected map $MSpin \to ko$
  - Atiyah-Bott-Shapiro, Anderson-Brown-Peterson
  - Upshot: in low degrees, we know the k-invariants

k-invariants for MSO and MU

- At $p = 2$, MSO is a wedge sum of shifts of Eilenberg-Mac Lane spectra: k-invariants vanish
- For odd primes (MSO) and all primes (MU), each is a sum of shifts of *Brown-Peterson spectra BP*
  - Relevant k-invariants determined in the original paper of Brown-Peterson
- Upshot: for MSO, only nonzero k-invariant below degree 8 is from π_0 to π_4, and is 3-primary

Stable homotopy theory is easy (easier) over $\mathbb{Q}$

- As we saw, stable homotopy operations are trivial over $\mathbb{Q}$
- So the AHSS is simpler over $\mathbb{Q}$
- But it turns out that all differentials and extension problems are trivial!
- Even stronger: the ∞-category of rational spectra is equivalent to the ∞-category of chain complexes over $\mathbb{Q}$

Thom spectra as a source of more examples

- If $V \to X$ is a virtual vector bundle, there is an associated *Thom spectrum* X^V
  - Important and rich theory that we don't have time to dig into
  - Uses: bordism, orientations and Poincaré duality for generalized cohomology theories, ...
- These give us further examples to play with in spectral sequences

$\mathscr{A}$-actions on Thom spectra

- The *Thom isomorphism theorem* yields an isomorphism of graded abelian groups $H^*(X;\mathbb{F}_2) \xrightarrow{\cong} \widetilde{H}^{*+\mathrm{rank}(V)}(X^V;\mathbb{F}_2)$
  - Not necessarily a ring structure on the cohomology of X^V
  - Cohomology classes for X^V are denoted Ux, where $x \in H^*(X;\mathbb{F}_2)$ and U has degree $\mathrm{rank}(V)$
- This is *not* an isomorphism of $\mathscr{A}$-modules!
- Instead, $\mathrm{Sq}(U) = w(V)$, and $\mathrm{Sq}(Ux)$ is formally evaluated with the Cartan formula

Generalized Thom isomorphisms

- The k-invariants of ko only depend on Sq^1 and Sq^2, so if V is spin, there should be no difference in the AHSSes for $ko^*(X)$ and $\widetilde{ko}^*(X^V)$
- Stronger theorem: generalized orientation theory gives conditions on V for the Thom isomorphism to hold in multiplicative generalized cohomology theories
  - $H\mathbb{Z}$: an orientation in the usual sense: $w_1(V) = 0$
  - KU and ku: a spin^c structure: $w_1(V) = 0$ and $\beta w_2(V) = 0$
  - KO and ko: a spin structure
  - G-bordism: a G-structure (if MG is a ring spectrum!)

Example: low-degree spin bordism of a Thom spectrum

- Goal: compute $\Omega_3^{\mathrm{Spin}}((B\mathbb{Z}/n)^{V-2})$, where V is the two-dimensional rotation representation
- For n odd, V is spin, so this reduces to $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/n)$
  - This is $\mathbb{Z}/n$, and this is sort of trivial
- For n even, V is not spin, and life is more interesting

Input data

- $H^*(B\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}[x]/(nx)$, $|x| = 2$
- $H^*(B\mathbb{Z}/n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x,y]/?$, where either $x^2 = 0$ $(n \neq 2)$ or $x^2 = y$ $(n = 2)$
- $\mathrm{Sq}(x) = x + x^2$, $\mathrm{Sq}(y) = y + y^2$
- $w_1(V) = 0$, $w_2(V) = y$

Calculation