Exploiting additional structure

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- Today's goal: structure $\implies$ computations
- First: some general stuff to this effect
- Then: an example involving Lyndon-Hochschild-Serre and twisted cohomology of S_4
- Then: interactions between homotopy fixed-point and Tate spectral sequences

Functoriality in the input data

- Serre SS: a map of fibrations; a map between coefficient groups
- Lyndon-Hochschild-Serre: a pullback of extensions; a map of coefficient groups
- Atiyah-Hirzebruch: maps between the spectra or between the spaces
- Common use: someone already computed something, and the case of interest maps to/from it

Pairings between two related spectral sequences

- The cap product pairing (aka *Kronecker pairing*) on cohomology induces pairings in the Serre and Atiyah-Hirzebruch spectral sequences
- That is: for each r we have a pairing $\langle -,-\rangle_r : E_r^{n,-s} \otimes E^r_{n,t} \to \pi_{s+t}(E)$ (where E is our spectrum of interest)
- For $r=2$ this is the pairing on co/homology
- The differentials are duals: $\langle d_r(-),-\rangle_r = \langle -, d^r(-)\rangle_r$
- The pairing on the E_∞ and E^∞ pages is compatible with the cap product pairing in E-cohomology

Module structures

- Serre and Atiyah-Hirzebruch SS: the homological SS is a module over the cohomological SS for the same data, arising from the cap product inducing an $H^*(X)$-module structure on $H_*(X)$
- If E is a ring spectrum and M is an E-module spectrum, the M-cohomology Atiyah-Hirzebruch SS is a module over the E-cohomology SS
- (Lyndon-Hochschild-)Serre: nontrivial local coefficients SS is a module over the SS with same input data and untwisted coefficients
- Later today, we'll see this structure in the homotopy fixed-points and Tate spectral sequences
- Occasionally these help you deduce differentials

Summary

- Compatible natural (functorial) structures in the E_2- and E_∞-pages of a spectral sequence tend to imply data of maps between the entire spectral sequences
- This is a lot of data, which is *good* because it constrains differentials

Extended example: twisted cohomology of S_4

- Consider the Lyndon-Hochschild-Serre spectral sequence for the extension $1 \to A_4 \to S_4 \to \mathbb{Z}/2 \to 1$
- And for the twisted cohomology group $\mathbb{Z}_{w_1(\rho)}$, where ρ is the representation of S_4 given by the symmetries of a tetrahedron
  - That is, orientation-preserving symmetries act on $\mathbb{Z}$ by 1, and orientation-reversing ones act by -1
- We want to know $H^*(S_4; \mathbb{Z}_{w_1(\rho)})$ for $* \leq 5$
  - There are differentials, and the usual tricks eliminate some but not all: use a section $\mathbb{Z}/2 \to S_4$; map the extension to $1 \to \mathrm{SO}(3) \to (3) \to \mathbb{Z}/2 \to 1$
  - The LHSSS has no multiplicative structure! We could despair, *or...*

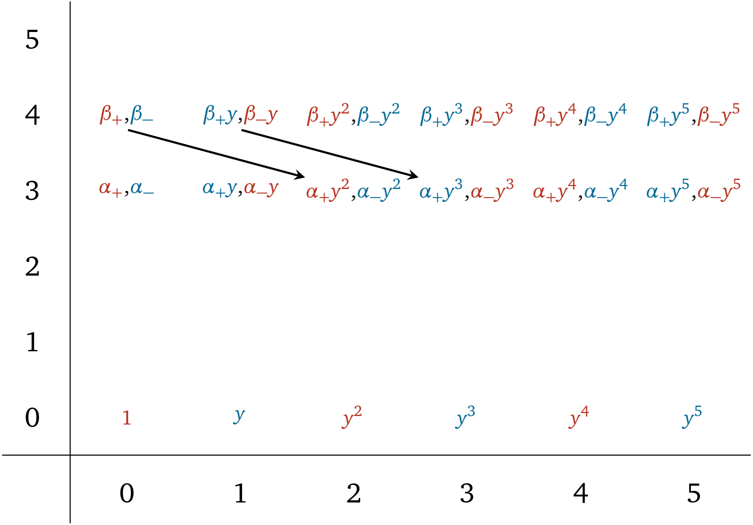
Compatibility with the untwisted cohomology LHSSS

- Let $R := \mathbb{Z}_{(2)}[x]/(x^2 - 1)$, which is a $\mathbb{Z}[\mathbb{Z}/2]$-module in which $\mathbb{Z}/2$ sends $x \mapsto -x$
- This is isomorphic to the $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathbb{Z} \oplus \mathbb{Z}_\sigma$, so we see both the twisted and the untwisted cohomology
- There's also a multiplicative structure! Compatible with the $\mathbb{Z}/2$-grading $(+,-)$, in which $\mathbb{Z}$ has grading $+$ and $\mathbb{Z}_\sigma$ has grading $-$
- Upshot: there is a multiplicative structure on the LHSSS for the cohomology with R coefficients
  - This is a *tri*graded spectral sequence, and that's OK $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2)$
  - Differentials are even in the $\mathbb{Z}/2$-grading

Input data

- Čadek computed $H^*(\mathbb{Z}/2; R) \cong \mathbb{Z}_{(2)}[y]/(2y)$, with $|y| = (1, -)$
- $H^*(A_4; \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}[\alpha, \beta, \dots]/(2\alpha, 2\beta, \dots)$ with $|\alpha| = 3$, $|\beta| = 4$, and all additional generators and relations in degrees 6 and above
- Upshot: $H^*(A_4; R) \cong \mathbb{Z}_{(2)}[\alpha_\pm, \beta_\pm, \dots]/(2\alpha_\pm, 2\beta_\pm, \dots)$, with $|\alpha_\pm| = (3, \pm)$ and $|\beta_\pm| = (4, \pm)$
- Thomas computes $H^*(S_4; \mathbb{Z}_{(2)})$: $H^4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$ and $H^5 \cong \mathbb{Z}/2$

The E_2-page


5
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Conclusion: the twisted cohomology groups

- $H^0 = \mathbb{Z}/2$
- $H^1 = \mathbb{Z}/2$
- $H^2 = 0$
- $H^3 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$
- $H^4 = \mathbb{Z}/2$
- $H^5 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

When are we ever gonna *use* this?

- This example came from a paper I'm working on!
- Goal: compute a group of isomorphism classes of invertible topological field theories with a particular tangential structure ($(BS_4; \rho)$-twisted spin structure)
- Adams spectral sequence tells us almost everything in degrees 7 and below, but there's a d_2 that the usual tricks fail to resolve
- To resolve it, compute the same information using an Atiyah-Hirzebruch SS
- This computation determines the input data to that AHSS