1. Steenrod squares

If you haven't seen Steenrod squares before, here are some introductory exercises.

(1) What is the $\mathcal{A}$-module structure on $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$?
(2) What is the $\mathcal{A}$-module structure on $H^*(\mathbb{CP}^n;\mathbb{Z}/2)$?
(3) How about $H^*(B\mathbb{Z}/n;\mathbb{Z}/2)$? Hint: use the Serre spectral sequence for the fibration $S^1 \to B\mathbb{Z}/n \to B\mathrm{U}_1 = \mathbb{CP}^\infty$ to understand the map $H^*(B\mathrm{U}_1;\mathbb{Z}/2) \to H^*(B\mathbb{Z}/n;\mathbb{Z}/2)$.
(4) Compute the $\mathcal{A}$-action on the cohomology of the Thom spectrum of the tautological bundle $\sigma \to \mathbb{RP}^n$. Geometrically, you might already know that the Thom *space* of this bundle is $\mathbb{RP}^{n+1}$; can you see that the Steenrod algebra actions match? (Note: there may be an off-by-one error due to our convention that Thom spectra are of virtual bundles in rank zero, i.e. $\sigma - 1$ instead of σ.)

2. Using Atiyah-Hirzebruch

(1) What is $K^*(\mathbb{RP}^2)$?
(2) Let L_n denote a lens space which is the quotient of S^3 by the $\mathbb{Z}/n$-action it receives as the unit sphere inside $\mathbb{C}^2$, where $\mathbb{C}^2$ carries the representation in which a generator of $\mathbb{Z}/n$ acts by $(e^{2\pi i/n}, e^{2\pi i/n})$. What are the (complex) K-theory groups of L_n? Can you generalize this to an arbitrary S^3/Γ, where Γ is a finite group of orientation-preserving diffeomorphisms acting freely on S^3? (Spoiler: this is worked out in https://arxiv.org/abs/1908.08027, appendix A.)
(3) For $k \le 5$, what is $\Omega_k^{\mathrm{Spin}}(B\mathrm{U}_1)$?
(4) If X is any space, show that the Atiyah-Hirzebruch spectral sequence for $\Omega_*^{\mathrm{SO}}(X)$ collapses in degrees 5 and below, i.e. if $p + q \le 5$, then differentials to or from $E_{p,q}^r$ vanish.
(5) Probably a little trickier: what can you say about $ko^k(B\mathbb{Z}/2)$ for $k \le 4$? For now, you should have one unresolvable extension problem left but can determine the remaining differentials.

3. A pinch of theory

(1) In this exercise, you'll prove that the first nonzero differential in the Atiyah-Hirzebruch spectral sequence is the k-invariant.
    (a) First, suppose E is a spectrum with exactly two nonzero homotopy groups. Then there can only be one differential in the Atiyah-Hirzebruch spectral sequence for E-theory. Show that it coincides with the k-invariant of E.
    (b) Now for a general choice of E, reduce to the previous case. Hint: every spectrum has an *n-connected cover* $\tau_{\ge n}E \to E$, which is an isomorphism on homotopy groups in degrees $\ge n$, but such that $\pi_k(\tau_{\ge n}E) = 0$ for $k < n$. Taking the cofiber of this map produces the $(n-1)$-*coconnected cover*[1] $E \to \tau_{<n}E$; this map induces an isomorphism on homotopy groups in degrees $n-1$ and below, and $\tau_{<n}E$ has no nonzero homotopy groups in degrees n and above.

[1] Occasionally jokingly called the $(n-1)$-nnected cover.