Chern-Simons Mini-course Notes

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1 Bundles, Connections, and Chern-Weil theory

Let G be a lie group.

1.1 Principal G Bundles

Definition. *A principal bundle on a manifold M is a manifold P with a surjective map $P \to M$ commuting with a free right G action, which acts transitively on fibers. A map of principal G bundles is a diffeomorphism $P \to P'$ commuting with the G actions and with the projections.*

Principal G bundles on a manifold M form a groupoid. As a first example, a principal G bundle on the point is just a manifold with free, transitive right G action, and is called a right G torsor. The groupoid of G torsors is equivalent to $*/G$. A general principal G bundle should be thought of as a bundle of G torsors.

1.2 Connections

The infinitesimal action of $\mathfrak{g}$ on a G bundle $\pi : P \to M$ gives an identification of each vertical tangent space with $\mathfrak{g}$. We thus have an exact sequence of vector bundles on P.

$$0 \to \underline{\mathfrak{g}} \to TP \to \pi^* TM \to 0$$

A connection is a splitting of this sequence which is invariant under the G action. A splitting is efficiently encoded as a one form $\Theta \in \Omega^1(P, \mathfrak{g})$ which restricts to the identity on the vertical. In other words Θ is the Maurer-Cartan form of G when restricted to any fiber. Invariance under G action is the following formula.

$$R_g^* \Theta = Ad_{g^{-1}} \Theta$$

If $s : M \to P$ is a section, one can pull down the connection to get a $\mathfrak{g}$ valued 1-form on M.

$$s^* \Theta \in \Omega^1(M, \underline{\mathfrak{g}})$$

This form determines Θ, and it is sometimes convenient to work with, but then you have to keep track of the dependence on s.

1.3 Curvature

A connection Θ defines (and is defined by) a distribution $ker(\Theta) \subset TP$. If this distribution is integrable, the connection is called flat. Non-flatness is measured by the frobeneous tensor of $ker(\Theta)$ which has a very nice formula, and is referred to as the curvature of Θ.

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$$

$\Omega \in \Omega^2(P, \mathfrak{g})$ vanishes on vertical tangent spaces, and satisfies the same transformation rule as Θ.

$$R_g^* \Omega = Ad_{g^{-1}} \Omega$$

This means we can view Ω as a 2-form on M valued in the adjoint bundle $\mathfrak{g}_P := P \times_G \mathfrak{g}$.

1.4 Chern-Weil Forms

In the 1940's, Chern and Weil found a homomorphism from the ring of invariant polynomials on $\mathfrak{g}$ to the real cohomology of BG.

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R})$$

When G is compact, their homomorphism is an isomorphism. If we have an invariant polynomial $f \in \mathbb{R}[\mathfrak{g}]^G$ of degree k, we can apply it to the curvature of a connection on a manifold M.

$$\omega_f := f(\Omega) \in \Omega^{2k}(M)$$

It turns out that the Chern-Weil form ω_f is closed, thus represents a cohomology class in $H^{2k}(M, \mathbb{R})$. Applying this construction to a connection on a universal bundle $EG \to BG$ gives the Chern-Weil homomorphism. For the purpose of Chern-Simons theory, we will only be using Chern-Weil 4-forms which are classified by invariant bilinear forms on $\mathfrak{g}$. In fact, we will always use bilinear forms which map to integral classes in $H^4(BG)$, and we will call such bilinear forms "integral". For example, the group of integral bilinear forms for $G = SU(n)$ is $\mathbb{Z}$ and is generated by

$$\langle X, Y \rangle = \frac{1}{8\pi^2} Tr(XY)$$

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1.5 Chern-Simons Invariants as Secondary Invariants

Let f be a degree k invariant polynomial as before. If M is a closed $2k$ manifold with connection (P, Θ), then we get an integer invariant of P.

$$\int_M \omega_f \in \mathbb{Z}$$

If M has non-empty boundary, this integrality will no longer hold, and the integral will depend on the connection. We use this to define a secondary invariant.

$$S_{CS}(\partial M; \Theta|_{\partial M}) := \int_M \omega_f \pmod{\mathbb{Z}}$$

A standard argument shows that S_{CS} doesn't depend on the extension of a connection to a $2k$ manifold.

1.6 Chern-Simons Forms

The main technical achievement of Chern and Simons was that for any invariant polynomial $f \in \mathbb{R}[\mathfrak{g}]^G$ of degree k, they constructed a an anti-derivative of ω_f in the following sense. For any connection Θ on a principal bundle $\pi : P \to M$, they constructed a $2k-1$ form α_f satisfying

$$d\alpha_f = \pi^* \omega_f$$

This allows one to express the Chern-Simons invariant S_{CS} in terms of

You can learn how to construct α_f in the exercizes if you wish, but for now we will content ourselves with the Chern-Simons 3-form. Let $\langle , \rangle$ be an invariant bilinear form on $\mathfrak{g}$. The Chern-Weil form is

$$\omega_f = \langle \Omega \wedge \Omega \rangle$$

. In the case when G is abelian, curvature is just $\Omega = d\Theta$, so the Chern-Weil form is $\pi^*\omega = \langle d\Theta \wedge d\Theta \rangle$, and one can (correctly) guess that that Chern-Simons form is $\alpha = \langle \Theta \wedge d\Theta \rangle$. The general formula is somewhat harder to guess.

$$\alpha = \langle \Omega \wedge \Theta \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle$$

1.7 Exercises

1. Let ω and α be the Chern-Weil 4-form and Chern-Simons 3 form for a given invariant inner product. Check that $d\alpha = \pi^*\omega$.

2. What bordism group must vanish for the definition of S_{CS} as a secondary invariant to work for any principal G bundle on any 3-manifold?

3. Show that the Chern-Weil forms $\omega_f(\Theta)$ and $\omega_f(\Theta')$ for two different connections on a principal bundle P are cohomologous. Do this by constructing a $2k-1$ form $\alpha_f(\Theta, \Theta')$ on M who's exterior derivative is the difference. Use the fiber integration formula (in the case $S = [0,1]$).

   Theorem. *If N is a manifold, and $\phi \in \Omega^k(N \times S)$ is a closed differential form, than there is the the following relation of fiber integrals.*

   $$d\int_{N\times S/N} \phi = \int_{N\times \partial S/N} \phi$$

4. Show that if $\pi : P \to M$ is a principal G bundle, π^*P has a cannonical trivialization.

5. Let Θ_0 be the trivial connection on π^*P coming from the cannonical trivialization in the previous problem. Let Θ be a connection on P, show that $\alpha(\Theta) := \alpha(\Theta_0, \pi^*\Theta)$ has what it takes to be a Chern-Simons 3-form. This is the construction of Chern and Simons.

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