Homological Algebra Summer Mini-course 2024: Lecture Notes

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July 14, 2024

Introduction

Theses are the lecture notes for the summer mini course on homological algebra I am running at UT Austin July 15th - 19th.

Contents

Chapter 1

Lecture 1

1.1 Administrative

1.1.1 Schedule

- Monday - Friday will look roughly like:
- Lecture: 9:30 - 11:00
- Office hours? (1:00 - 2:00)
- Problem Sessions: 2:00 - 4:00

1.1.2 Aims

1. You learn stuff
2. I teach stuff

1.1.3 On Lectures

- Text = Weibel - An Introduction to Homological Algebra.
- Notes can be found on my website: `https://lachlanpotter.github.io/` or the summer mini course discord server.
- Lectures 1-3: Chain complexes and abelian categories.
- Lecture 4: Projective and injective objects.
- Lecture 5: Derived functors, Tor and Ext.
- Everything is optional, but I hope you will come.

1.1.4 On Problem Sessions

- I have a way I envisage problem sessions running, I will say more about it in the afternoon.
- Please do whatever gives you the most benefit.
- I have prepared problems that are tied closely to lecture content, but feel free to practice whatever/however you like. In particular, any exercise in Weibel is probably good.
- There is no need to stay for the full two hours. As long as there are people doing math I will make myself available from 2:00-4:00.

1.2 Introduction

1.2.1 History of Homological Algebra

Disclaimer: I'm not a math historian (regrettably). The following is a summary of part the Intro to Weibel's fantastic book.

1890-1940: Ideas started brewing, mostly in the minds of algebraic Topologists, crystallising in $H_n(X), H^n(X;R)$.

1940 - 1956: People realise that the formalism can be applied to algebraic objects, birthing: derived functors R^iF/L_iF and injective/projective modules, which we will see in lecture.

1956 - 1975: further generalisations emerge: cohomology of sheaves emerge as foundational in algebraic geometry, the derived category emerges as an algebraic analogue of the topologist's homotopy category.

1.2.2 Motivation

Homological Algebra is a machine for building invariants of objects: topological spaces, modules, groups, lie algebras, sheaves.

The general slogan is that "left-right exact functors should be seen as the beginning of a long exact sequence". For example $-\otimes B$ is right exact, and continues to the left via the higher Tor groups, while $\mathrm{Hom}(A,-)$ is left exact and continues to the right via higher Ext groups.

This can be viewed as an algebraic analogue of the LES of (co)homology of topological spaces.

1.2.3 Course Plan

Define Tor and Ext $\leftrightsquigarrow$ Chapters 1 & 2 of Weibel.

This course has the modest aim of stating the definition of Tor and Ext for R-modules. This corresponds roughly to Chapter 1 & 2 of Weibel, with some parts cut out.

In doing so we will develop the formalism that will allow one to (more) easily read about the following:

- Cohomology of Sheaves.
- (Co)homology of groups.
- (Co)hmology of Lie algebras.

Chain Complexes & $\overset{1-3}{\text{homology}}$ + Abelian categories → injective and $\overset{4}{\text{projective}}$ modules → Derived $\overset{5}{\text{functors}}$

1.3 Chain Complexes

- Chain complexes, Homology
- Morphisms of Chain Complexes, Quasi-isomorphisms.

1.3.1 Chain Complexes

Definition 1.1. A chain complex of R-modules is a sequence:

$$C_\bullet = (\cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \to \cdots)$$

such that $d_n \circ d_{n+1} = 0$.

Notation 1.2. We define the following notation

- d_n are the differentials of $C_\bullet$.
- $Z_n := Z_n(C_\bullet) := \ker(d_n)$ the n-cycles.
- $B_n := B_n(C_\bullet) := \mathrm{im}\,(d_{n+1})$ the n-boundaries.
- $H_n := H_n(C) := Z_n/B_n = \ker(d_n)/\mathrm{im}\,(d_{n+1})$
- $C_\bullet$ is exact at C_n if $H_n = 0$, i.e. $Z_n = B_n$

The constraint that $d^2 = 0$ implies that $B_n \subset Z_n \subset C_n$ for all n.

Examples 1.3. If X is a simplicial or CW complex (as appropriate), then $C_\bullet^\Delta(X), C_\bullet^{sing}(X), C_\bullet^{CW}(X)$ are a chain complexes of abelian groups. The homology of these complexes give simplicial/singular/cellular homology classes of X. The elements of Z_n are cycles and the elements of B_n are boundaries.

Definition 1.4. A co-chain complex of R-modules is a sequence:

$$C^\bullet = (\cdots \to C^{n-1} \overset{d^{n-1}}{\to} C^n \overset{d^n}{\to} C^{n+1} \to \cdots)$$

such that $d^n \circ d^{n-1} = 0$. Essentially, a chain complex with the arrows reversed.

Remark 1.5. We make analogous definitions with appropriate superscripts. The indices of arrows are always determined by the domain.

Example 1.6. For any R-module, applying the functor $\mathrm{Hom}(-, M)$ to any chain complex yields a co-chain complex. Applying this to Example 1.3 yields familiar cohomology co-chain complexes.

If X is a smooth manifold, the k-differential forms on X naturally assemble into a co-chain complex, with differentials given by $d^k : \Omega^k(X) \to \Omega^{k+1}(X)$. The cohomology yield de Rahm cohomology.

1.3.2 Morphisms of Chain Complexes

We wish to form a category of chain complexes, for which we must define morphisms.

Definition 1.7. A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ is a collection of maps $f_n : C_n \to D_n$ which assemble into a commutative ladder diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d^C_{n+2}} & C_{n+1} & \xrightarrow{d^C_{n+1}} & C_n & \xrightarrow{d^C_n} & C_{n-1} & \xrightarrow{d^C_{n-1}} & \cdots \\ & & \downarrow{f_{n+1}} & & \downarrow{f_n} & & \downarrow{f_{n-1}} & & \\ \cdots & \xrightarrow{d^D_{n+2}} & D_{n+1} & \xrightarrow{d^D_{n+1}} & D_n & \xrightarrow{d^D_n} & D_{n-1} & \xrightarrow{d^D_{n-1}} & \cdots . \end{array}$$

We often abbreviate the above to the equality $df = fd$

Exercise 1.8. Prove that a map of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ induces a map $f_* : H_*(C_\bullet) \to H_*(D_\bullet)$

Note 1.9. Demonstrate the diagram chase above

Definition 1.10. A map of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if it induces an isomorphism $H_n(C_\bullet) \to H_n(D_\bullet)$ for all $n \in \mathbb{Z}$.

1.4 Operations on Chain Complexes.

- shifting, direct sums, products, kernels, cokernels, exactness.

Definition 1.11. Given a complex $C_\bullet$ we can define $(C_\bullet[p])_n := C_{n+p}$, and differentials given by $(-1)^p d$. The sign convention is to simplify later notation.

For co-chain complexes we define $(C^\bullet[p])_n := C_{n-p}$.

Slogan 1.12. Positive shifts move things to the right.

Definition 1.13. Given two complexes $(C, d^C), (B, d^B)$, we can form the direct sum $(C \oplus B, d^{C \oplus B})$ as:

$$\cdots \to C_n \oplus B_n \overset{d^C \oplus d^B}{\to} C_{n-1} \oplus B_{n-1} \to \cdots$$

where $d^C \oplus d^B : C_n \oplus D_n \to C_{n-1} \oplus B_{n-1}$, $c + b \mapsto d^C(c) + d^B(b)$. This will easily generalise to direct sums over an arbitrary index set.

Exercise 1.14 (Weibel page 5)**.** For a family of chain complexes $\{C_i\}_{i\in I}$ define the product of chain complexes $\prod_{i\in I} C_i$.

Definition 1.15. A complex $B_\bullet$ is a subcomplex of $C_\bullet$ if B_n is a submodule of C_n for all n, and the differentials are the restriction of the differentials on $C_\bullet$.

I.e. the inclusionsn($i_n : B_n \hookrightarrow C_n$) assemble into a chain map $B_\bullet \to C_\bullet$.

Definition 1.16. Given a subcomplex $B_\bullet \subset C_\bullet$. We define the quotient complex $(C_\bullet/B_\bullet)_n := C_n/B_n$, with differentials induced from the differentials of C.

Note 1.17. Demonstrate the diagram chase for $C_\bullet/B_\bullet$

Example 1.18. If $f_\bullet : B_\bullet \to C_\bullet$ is a map of chain complexes, then $\{\ker(f_n)\}_{n\in\mathbb{Z}} \subset B_\bullet$ and $\{\mathrm{coker}(f_n)\}_{n\in\mathbb{Z}}$ assemble into a quotient complex of C. We call these complexes $\ker(f_\bullet)$ and $\mathrm{coker}(f_\bullet)$ respectively

Exercise 1.19. Check that the differentials for $\ker(f_\bullet)$ and $\mathrm{coker}(f_\bullet)$ are well-defined, and that $d^2 = 0$.

Definition 1.20. One can similarly define $\mathrm{im}\,(f_\bullet)$, or alternatively take the definition to be $\ker(C_\bullet \to \mathrm{coker}(f_\bullet))$.

Definition 1.21. A sequence of chain complexes $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ is exact if $\ker(f_\bullet) = 0$, $\ker(g_\bullet) = \mathrm{im}\,(f_\bullet)$ and $\mathrm{coker}(g_\bullet) = 0$.

Equivalently $0 \to A_n \to B_n \to C_n \to 0$ is exact for all $n \in \mathbb{Z}$.

1.5 Long Exact Sequence of Homology

- The Snake Lemma
- LES of Homology

If you have studied algebraic topology you know how useful it can be to have a long exact sequence associated to a short exact sequence. The aim of this section is to prove the following theorem.

Theorem 1.22. *Let* $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{b} C_\bullet \to 0$ *be a short exact sequence of complexes. Then there are natural maps* $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ *forming a long exact sequence:*

$$\cdots \xrightarrow{g} H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f} H_n(B_\bullet) \xrightarrow{g} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{g} \cdots .$$

furthermore, $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ *is defined by:*

$$f_n(b) + d_{n+1}(C_{n+1}) \mapsto a + d_n(A_n), \text{ where } f'_{n-1}(a) = d_n(b).$$

Note 1.23. Mention that the precise notion of naturality is spelled out in the notes and in Weibel Prop 1.3.4.

Remark 1.24. Natural here means something precise. In particular if one has a morphism of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{f} & B_\bullet & \xrightarrow{g} & C_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{f'} & B'_\bullet & \xrightarrow{g'} & C'_\bullet & \longrightarrow & 0 \end{array}$$

then this gives rise to a commutative ladder diagram of long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(B_\bullet) & \xrightarrow{g} & H_n(C_\bullet) & \xrightarrow{\partial} & H_{n-1}(A_\bullet) & \xrightarrow{f} & H_{n-1}(B_\bullet) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(B'_\bullet) & \xrightarrow{g'} & H_n(C'_\bullet) & \xrightarrow{\partial'} & H_{n-1}(A'_\bullet) & \xrightarrow{f'} & H_{n-1}(B'_\bullet) & \longrightarrow & \cdots \end{array}$$

of course due to Exercise 1.8, the only commutative squares requiring proof are the ones involving ∂.

The proof of this theorem relies on the snake lemma.

Lemma 1.25 (The Snake Lemma)**.** Consider a commutative diagram of R-modules:

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{i} & B' & \longrightarrow & C' & & \end{array}$$

Then there exists an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0$$

where ∂ is given by the formula $\partial(c) := i^{-1}gp^{-1}(c)$, $c \in \ker(h)$.

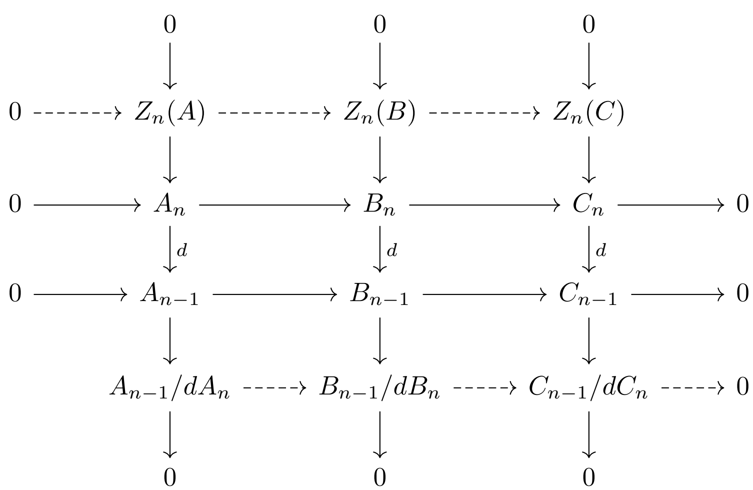
Proof. go over the construction of the map, extending the diagram with kernels and cokernels. Mention that there remains to prove the following:

- well definedness of the map (there was a choice made at the p^{-1} step).
- exactness at the points involving ∂.

□

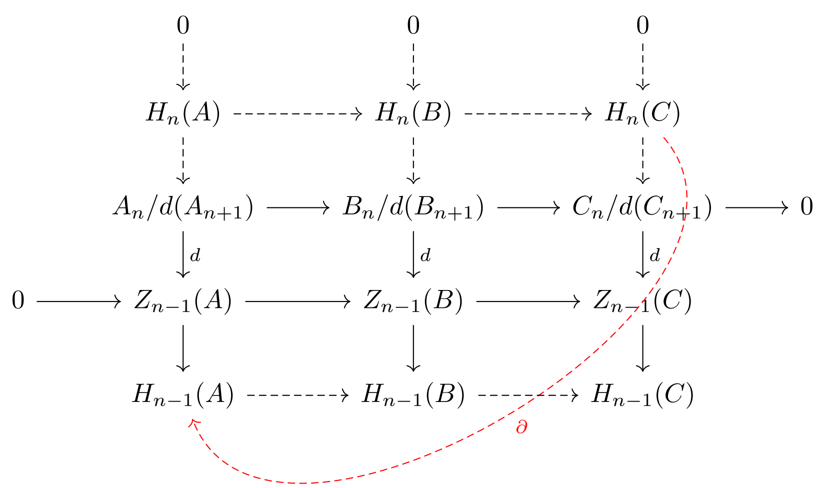
We are now ready to prove the main theorem:

Proof. Consider just one differential forming the exact sequence of chain complexes, and extend at the start and end with kernels and cokernels:



a diagram chase yields the existence of the dashed lines, and their exactness.

Now taking the exactness of the above rows, we get a diagram of solid arrows, whose kernels and cokernels are the homology groups we are looking for. Applying the snake lemma we get the following dashed arrows.



□

Chapter 2

Lecture 2

2.1 Formalism of Abelian Categories

- Ab $\subset$ additive categories.
- Monos, epis, kernels & cokernels.
- Examples in R-**Mod** and $\mathbf{Ch}(\mathcal{A})$.
- Abelian categories.
- Familiar notions in Abelian Categories.

The aim of this section is to define abelian categories, which are an abstraction the category of R-modules, which take the parts necessary to be able to perform homological algebra.

Note 2.1. This lecture is very abstract, so if it is all too much, just try to hold on until the Freyd-Mitchell Embedding theorem in the next lecture.

2.1.1 Ab and Additive Categories

To hone in on the correct definition we have three type of categories, which form more and more specific objects.

$$\{\mathbf{Ab}\text{-categories}\} \supset \{\text{additive categories}\} \supset \{\text{abelian categories}\}$$

Definition 2.2. A category $\mathcal{A}$ is an **Ab**-category if every Hom set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, in such a way that the group operation is compatible with function composition. I.e. for any diagram:

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g' + g)f = hg'f + hgf$.

Definition 2.3. An additive category is an **Ab**-category with a zero object 0 (i.e. a single object which is terminal and initial) and such that all binary products exist.

Examples 2.4. The category of non-zero abelian groups is an **Ab**-category which is not additive.

The category $\mathbf{Ch}(R{-}\mathbf{Mod})$ is an additive category, with the zero chain complex the obvious choice and the product of chain complexes defined componentwise.

Remark 2.5. Just the structure of an additive category means that finite coproducts exist, and are isomorphic to the corresponding products. (Take this as an exercise if you wish). This mirrors the behaviour we see in R-mod.

2.1.2 Monos, Epis, Kernels and Cokernels

First we generalise the notions of injective and surjective.

Definition 2.6. A monomorphism, or a monic is a morphism $i : A \to B$ such that $ig = 0 \implies g = 0$ for all $g : A' \to A$.

An epimorphism, or an epi is a morphism $\pi : A \to B$ such that $g\pi = 0 \implies g = 0$ for all $g : B \to B'$.

Lemma 2.7. In R-Mod, a morphism $f : A \to B$ is epi iff it is surjective.

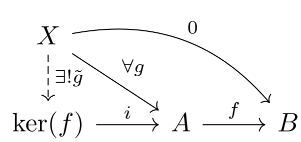
Proof. ($\impliedby$) : this direction is clear because for all $b' \in B$, $b' = \pi(b)$ and therefore $g(b') = g\pi(b) = 0$.

($\implies$) : Consider $g : B \to B/\pi(A)$, then clearly $g\pi = 0$, but since π is epi, then $g = 0$ which implies that $B = \pi(A)$, so π is surjective. $\square$

Exercise 2.8. Prove that in R-**Mod** , a morphism $f : A \to B$ is monic iff it is injective.

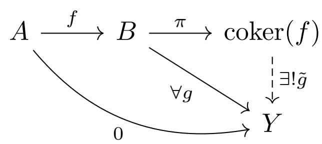
Taking kernels and cokernels are fundamental operations when working in R-mod, here we give a categorical definition that will work in the abstract setting.

Definition 2.9. Let $f : A \to B$ be a morphism in an additive category $\mathcal{A}$. The kernel of f, is the data $(K, K \to A)$ of an object $K \in \mathcal{A}$ **and a morphism** $K \xrightarrow{i} A$. Satisfying the universal property that for all maps $g : X \to A$ such that $f \circ g = 0$ there exists a **unique** map $\tilde{g} : X \to K$ such that $g = i \circ \tilde{g}$. In a diagram:



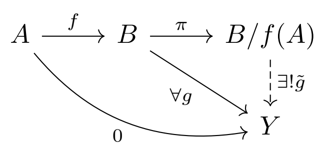
Remark 2.10. While often denote the object K by $\ker(f)$ it is important to remember that the kernel is actually the object *and* the morphism $\ker(f) \to A$. This data together determines an object up to unique isomorphism, justifying the terminology "the" kernel.

Definition 2.11. Let $f : A \to B$ be a morphism in an additive category $\mathcal{A}$. The cokernel of f, is the data $(\text{coker}(f), B \to \text{coker}(f))$ satisfying the following universal property.



Lemma 2.12. Kernels and cokernels exist in R-**Mod** and coincide with the usual definition of kernel/cokernel, along with their canonical maps.

Proof. Let $f : A \to B$ be a morphism and consider $\pi : B \to B/f(A)$. We wish to show that this map is a cokernel in the sense of Definition 2.11. Indeed suppose $g : B \to Y$ is such that $gf = 0$, then certainly $g(f(A)) = 0$, hence by the universal property of the quotient, g descends to a unique map $\tilde{g}$:



□

Exercise 2.13. Prove the case of kernels.

Exercise 2.14. If $f_\bullet : A_\bullet \to B_\bullet$ is a map of complexes, then $\ker(f_\bullet)$ and $\text{coker}(f_\bullet)$ are kernels and cokernels respectively in the sense of Definition 2.11

2.1.3 Abelian Categories

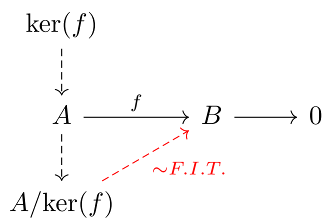
We are now ready for the big definition of this section. We line up each of these axioms with the corresponding fact in R-**Mod** for concreteness.

Definition 2.15. An abelian category is an additive category such that:

- All kernels and cokernels exist.
- Every monic is the kernel of its cokernel.

$$0 \longrightarrow \underset{=\ker(\pi)}{A} \xrightarrow{f} B \overset{\pi}{\dashrightarrow} B/f(A)$$

- Every epi is the cokernel of its kernel.



Examples 2.16.

- $\mathbf{Mod}_R$ is an abelian category.
- $\mathbf{Ch}(\mathbf{Mod}_R)$ is an abelian category, Exercise 2.14 is the first axiom.
- If $\mathcal{A}$ is an abelian category, then $\mathbf{Ch}(\mathcal{A})$ is also an abelian category.
- $\mathbf{AbShv}_X$, the category of sheaves of abelian groups on a topological space is also an abelian category, there is also something to prove.

2.1.4 Familiar Notions

Definition 2.17. In an abelian category, define $\operatorname{im}(f) := \ker(\operatorname{coker}(f))$. Keep in mind the construction of the cokernel for abelian groups.

Remark 2.18. We finally have defined enough concepts that **all definitions made in lecture 1 now make sense *essentially verbatim* in any abelian category**. A sequence of morphisms in an abelian category is a chain complex if $d \circ d = 0$, a complex is exact if $\ker = \operatorname{im}$ at every point, homology $= \ker/\operatorname{im}$ etc...

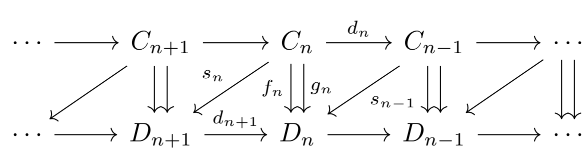
2.2 Chain Homotopies

- Chain Homotopies & induced map on homology
- Split Complexes

2.2.1 Chain Homotopies

Definition 2.19. We say that two chain maps $f, g : C \to D$ are homotopic, denoted $f \sim g$ if there exist maps $s_n : C_n \to D_{n+1}$ such that $f - g = sd + ds$. A map f is null-homotopic if $f \sim 0$.

This can be drawn as a **non-commutative** diagram:



such that the difference between the two vertical arrows is equal to the sum of the two other paths straight down.

Lemma 2.20. If $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic, then they induce the same maps on homology.

Proof. By considering $f - g$ we may assume that $g = 0$, that is $f = sd + ds$.

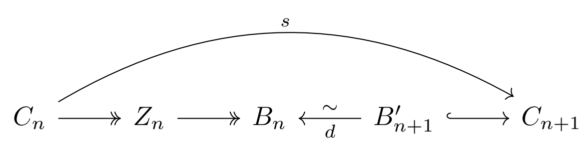
Choose any element of $H_n(C)$, represented by a cycle $c \in Z_n(C)$. Then its image in $H_n(D)$ is represented by $f(c) = sd(c) + d(sc) = d(sc) \in B_n(D)$. Therefore $f(c)$ is zero in $H_n(D)$. □

2.2.2 Split Complexes

Suppose we have a complex of vectorspaces $C_\bullet$. For any subspace $U \subset V$ there is an isomorphism $V \cong U \oplus V/U$, using this we gain the following decompositions:

$$C_n = Z_n \oplus B'_n, \qquad B'_n \cong C_n/Z_n \underset{d}{\overset{\sim}{\to}} d(C_n) = B_{n-1}$$
$$Z_n = B_n \oplus H'_n, \qquad H'_n \cong Z_n/B_n = H_n(C_\bullet)$$

Using these decompositions, we construct the following map:



These maps satisfy $dsd = d$. Whence the following definition.

Definition 2.21. A chain complex is called split of there exist maps $s_n : C_n \to C_{n+1}$ such that $dsd = d$.

Exercise 2.22. Prove that an exact sequence is split if and only if there are decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$.

Chapter 3

Lecture 3

3.1 Fundamental Results on Abelian Categories

- Additive functors and exactness.
- Freyd-Mitchell Embedding Theorem and consequences.
- Left exactness of Hom
- Yoneda for additive functors.

3.1.1 Additive Functors and Exactness

Just as abelian categories are categories with extra structure. The correct notion of morphism of categories are functors respecting that structure.

Definition 3.1. A functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the maps $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ are group homomorphisms.

Essentially all reasonable functors between abelian categories will be additive, so this adjective is often dropped in theorem statements.

Definition 3.2. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is left exact if for all exact sequences $0 \to A \to B \to C \to 0$ we have that $0 \to FA \to FB \to FC$ is exact.

Remark 3.3. The definitions of exact and right exact functors are left to the reader.

If $\mathcal{A}$ is an abelian category then so is $\mathcal{A}^{\mathrm{op}}$, a left exact contravariant functor is one that maps exact sequences to left exact sequences. I.e. $(0 \to A \to B \to C \to 0) \mapsto (0 \to FC \to FB \to FA)$.

3.1.2 Freyd-Mitchell Embedding Theorem

Theorem 3.4 (Freyd-Mitchell Embedding)**.** *If $\mathcal{A}$ is a small abelian category, then there is a ring R and an exact, fully faithful functor $\iota : \mathcal{A} \to \mathbf{Mod}_R$, which embeds $\mathcal{A}$ as a full subcategory in the sense that $\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_{\mathbf{Mod}_R}(\iota M, \iota N)$.*

If you know what a fully faithful functor means great, if not the following lemma will suffice for understanding.

Lemma 3.5 (Weibel Lemma 1.6.2)**.** If $\mathcal{C} \subset \mathcal{A}$ is a full sub-category of an abelian category $\mathcal{A}$.

1. $\mathcal{C}$ is additive $\iff$ $0 \in \mathcal{C}$, and $\mathcal{C}$ is closed under $\oplus$.
2. $\mathcal{C}$ is abelian and $\mathcal{C} \subset \mathcal{A}$ is exact $\iff$ $\mathcal{C}$ is additive and closed under ker and coker.

Since it may be unclear how to use this theorem in practice, we formulate some corollaries.

Corollary 3.6 (Corollaries of Freyd-Mitchell)**.** *Let $\iota : \mathcal{A} \to \mathbf{Mod}_R$ be the embedding given by the Freyd-Mitchell embedding theorem.*

- *A sequence is exact iff it is exact after applying ι.*
- *A morphism $A \to B$ exists iff there exists a morphism $\iota(A) \to \iota(B)$ in $\mathbf{Mod}_R$.*
- *A diagram in $\mathcal{A}$ commutes iff its image commutes in $\mathbf{Mod}_R$.*

Corollary 3.7 (The snake lemma in an abelian category)**.** *Consider a commutative diagram in any abelian category $\mathcal{A}$:*

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{i} & B' & \longrightarrow & C' & & \end{array}$$

Then there exists an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{coker}(h) \to 0$$

Proof. Take the small abelian subcategory $\mathcal{C}$ of $\mathcal{A}$ generated by the objects in the diagram. Then by the snake lemma for R-Modules there is a map ∂ and an exact sequence in $\mathbf{Mod}_R$. Therefore there is a map and a sequence in $\mathcal{C}$ which is exact, and therefore exact in $\mathcal{A}$. $\square$

3.1.3 Properties of Hom

We finish up with a proposition, whose proof can be made psychologically easier using the Freyd-Mitchell Embedding Theorem.

Proposition 3.8 (Left exactness of Hom)**.** *Let $M \in \mathcal{A}$ be an object in an abelian category, then $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathbf{Ab}$ is a (covariant) left-exact functor.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be exact, we want to show that:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(M, C)$$

is exact. We show that $\ker(g_*) \subset \mathrm{im}\,(f_*)$. By the Freyd-Mitchell embedding Theorem, we may assume that A, B, C, M are R-Modules.

Suppose that $(\phi : M \to B) \in \ker(g_*)$, then $g_*(\phi) = g \circ \phi = 0$. Therefore $\mathrm{im}\,(\phi) \subset \ker(g) = \mathrm{im}\,(f)$ by exactness of the original sequence. Since f is injective this means that there exists $\tilde{\phi} : M \to A$ such that $f_*(\tilde{\phi}) = f \circ \tilde{\phi} = \phi$. Therefore $\phi \in \mathrm{im}\,(f_*)$, as required. □

Exercise 3.9. Prove the remaining parts of the above proposition.

3.1.4 Yoneda Lemma for Abelian Categories

Theorem 3.10 (The Yoneda lemma for abelian categories)**.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of morphisms in an abelian category $\mathcal{A}$. If for all objects $M \in \mathcal{A}$ the sequence:*

$$\mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(M, C)$$

is exact, then the original sequence is exact.

Proof. Since the proof will only require a choice of finitely many M we may assume everything is happening in $\mathbf{Mod}_R$.

$gf = 0$: Let $M = A$, then $gf = g_*(f_*(\mathrm{id}_A)) = 0$.

$\ker(g) \subset \mathrm{im}\,(f)$: Let $M = \ker(g)$ and $(i : \ker(g) \to B) \in \mathrm{Hom}_{\mathcal{A}}(\ker(g), B)$. Then $g_*(i) = g \circ i = 0$. Therefore $i \in \ker(g_*) = \mathrm{im}\,(f_*)$, that is, there exists some $\tilde{i} : \ker(g) \to A$ such that $f \circ \tilde{i} = i$. Therefore $\ker(g) = \mathrm{im}\,(i) \subset \mathrm{im}\,(f)$. □

3.2 Adjoint functors

- Adjoints in general.
- Adjoints and exactness.

3.2.1 Adjoint Functors in General

Slogan 3.11. Adjoint functors appear everywhere.

Definition 3.12. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be two functors. We say L is left adjoint to R, and R is right adjoint to L or simply $L \dashv R$ if either of the two equivalent definitions hold:

1. There exists for all $C \in \mathcal{C}, D \in \mathcal{D}$ an isomorphism(bijection):

$$\tau_{C,D} : \mathrm{Hom}_{\mathcal{D}}(LC, D) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(C, RD)$$

   That is natural in C and D, meaning that for $f : C' \to C$ and $g : D \to D'$ the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(LC, D) & \xrightarrow{\tau_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, RD) \\ {\scriptstyle -\circ L(f)}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \mathrm{Hom}_{\mathcal{D}}(LC', D) & \xrightarrow{\tau_{C,D'}} & \mathrm{Hom}_{\mathcal{C}}(C', RD) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(LC, D) & \xrightarrow{\tau_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, RD) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle R(g)\circ -} \\ \mathrm{Hom}_{\mathcal{D}}(LC, D') & \xrightarrow{\tau_{C,D'}} & \mathrm{Hom}_{\mathcal{C}}(C, RD') \end{array}$$

2. There exists two natural transformations (called the counit and unit respectively)

$$\varepsilon : LR \implies 1_{\mathcal{D}} \text{ and } \eta : id_{\mathcal{D}} \implies RL$$

   such that 1_L the following diagrams of natural transformations commute:

$$\begin{array}{ccc} L & \overset{L\eta}{\Longrightarrow} & LRL \\ & {\scriptstyle 1_L} \searrow\!\!\searrow & \Downarrow{\scriptstyle \varepsilon L} \\ & & L \end{array} \qquad \begin{array}{ccc} R & \overset{\eta R}{\Longrightarrow} & RLR \\ & {\scriptstyle 1_R} \searrow\!\!\searrow & \Downarrow{\scriptstyle R\varepsilon} \\ & & R \end{array}$$

Remark 3.13. The equivalence of these two definitions should not be obvious, but sometimes one is more useful than the other to check/use. In practice most people simply define τ or ε and η in a way that seema "canonical enough" and call it a day. We will use defintion 1 more commonly.

Remark 3.14. One extremely common place for adjoints to appear are what I call "creation-deletion" adjoint pairs, In this case the right adjoint is commonly some form of "forgetful" functor, while the left adjoint is some sort of "free" object.

Examples 3.15.

- $U : \mathbf{Vect}_k \to \mathbf{Set}$ has left adjoint to $R : \mathbf{Set} \to \mathbf{Vect}, S \mapsto k^S$
- $U : \mathbf{Ring} \to \mathbf{Set}$ has left adjoint to $R : \mathbf{Set} \to \mathbf{Ring}, S \mapsto \mathbb{Z}[S]$
- $U : \mathbf{Ring} \to \mathbf{Ab}$ has left adjoint to $T : \mathbf{Ab} \to \mathbf{Ring}, A \mapsto T(A)$ where $T(A)$ is the tensor algebra of A.

- $U : \mathbf{Mod}_R \to \mathbf{Ab}$ has left adjoint to $R \otimes_{\mathbb{Z}} - : \mathbf{Ab} \to \mathbf{Mod}_R$.
- $U : \mathbf{Shv} \to \mathbf{PreShv}$ has left adjoint to the sheafification functor $\mathbf{PreShv} \to \mathbf{Shv}$.

The most important adjunction pair for the purposes of this course, is the so called Tensor-Hom adjunction.

Proposition 3.16. *Let B be an R-S bimodule, then $- \otimes_R B : \mathbf{Mod}_R \to \mathbf{Mod}_S$ is left adjoint to $\mathrm{Hom}_S(B, -) : \mathbf{Mod}_S \to \mathbf{Mod}_R$.*

Proof. Given $A \in \mathbf{Mod}_R$ and $C \in \mathbf{Mod}_S$ morphisms $\tau_{A,C}$ are given by:

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)), \tau f(a) : b \mapsto f(a \otimes b)$$

To define an inverse take $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$, notice that $(a, b) \mapsto g(a)(b)$ is an R-bilinear map $A \times B \to C$, so it makes sense to define:

$$\eta g : A \otimes_R B \to C,\ a \otimes b \mapsto g(a)(b)$$

The requisite checks are left to the reader. □

3.2.2 Adjoints Functors Between Abelian Categories

A useful corollary is the following:

Corollary 3.17. *Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be a left-right adjoint pair of additive functors between abelian categories. Then L is right-exact and R is left exact.*

Proof. Consider an exact sequence $0 \to B' \to B \to B'' \to 0$. Then to check left exactness of $RB' \to RB \to RB''$ it's sufficient by the Yoneda lemma to check for all $A \in \mathcal{A}$ the induced sequence of Hom-groups is, but we have:

$$\begin{array}{ccccccc} & & \mathrm{Hom}_{\mathcal{A}}(A, RB') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, RB) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, RB'') \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(LA, B') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(LA, B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(LA, B'') \end{array}$$

Where the bottom row is left exact because $\mathrm{Hom}(LA, -)$ is left exact by the left exactness of Hom. Since this holds for all $A \in \mathcal{A}$, $0 \to RB' \to RB \to RB''$ is exact. □

This gives us a slick proof of an essential fact we will use later:

Corollary 3.18. *Let B be an R-module, $- \otimes_R B : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right exact, and $\mathrm{Hom}_R(B, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is left exact.*

Chapter 4

Lecture 4

4.1 Projective Objects

- Projectives in $\mathbf{Mod}_R$.
- Projective resolutions.
- Comparison theorem.
- Horseshoe lemma.

4.1.1 Projective Objects

Definition 4.1. We call an object $P \in \mathcal{A}$ projective, if for every morphism $\gamma : P \to C$ and epimorphism $\pi : B \to C$, there exists at least one $\tilde{\gamma} : P \to B$ such that $\pi\tilde{\gamma} = \gamma$. This is often written diagrammatically:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\exists\tilde{\gamma}} & \downarrow{\scriptstyle\gamma} & & \\ B & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

Remark 4.2. Equivalently, projective objects are the objects such that $\mathrm{Hom}(P, -)$ is an exact functor.

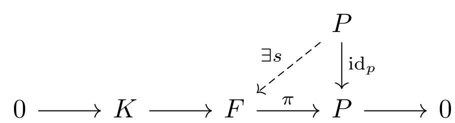
There is no uniqueness in the above definition, so there can be many non-isomorphic projective objects.

Note 4.3. Notice that free modules are projective in $\mathbf{Mod}_R$.

We can in fact classify the projective objects in $\mathbf{Mod}_R$.

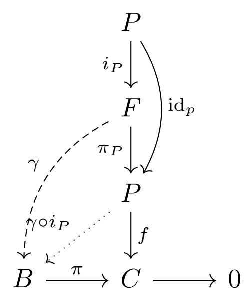
Proposition 4.4. *An R-Module is projective iff it is a direct summand of a free R-Module.*

Proof. ($\implies$) : Let P be projective. Then choose a free module F that surjects onto P, this yields a short exact sequence:



These exists a splitting s by the projectivity of P, so by the splitting lemma $F \cong P \oplus F$, as required.

($\impliedby$) : Let $F = P \oplus K$ be free, we wish to show that P is projective. Let $p : B \to C$ be an epi and chase the following diagram to see that P is projective.



∎

We prove here briefly a corollary that will be critically important later.

Corollary 4.5. *Projective modules are flat.*

Proof. Let P be projective. Then there is a free module F such that $F \cong P \oplus K$. To prove that P is flat we need only show that $\otimes P$ preserves injective maps. Let $0 \to A \to B$ be injective, since F is free, hence flat, we have the following diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & F \otimes A & \longrightarrow & F \otimes B \\
 & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & (P \otimes A) \oplus (F \otimes A) & \longrightarrow & (P \otimes B) \oplus (F \otimes B) \\
 & & \uparrow & & \uparrow \\
 & & P \otimes A & \longrightarrow & P \otimes B
\end{array}$$

Chasing this diagram tells us that $P \otimes A \to P \otimes B$ is injective. ∎

Remark 4.6. It is tempting to think that projective modules are exactly the free modules, but this is not always the case. For example if $R = R_1 \times R_2$ or $R = M_n(F)$.

4.1.2 Projective Resolutions

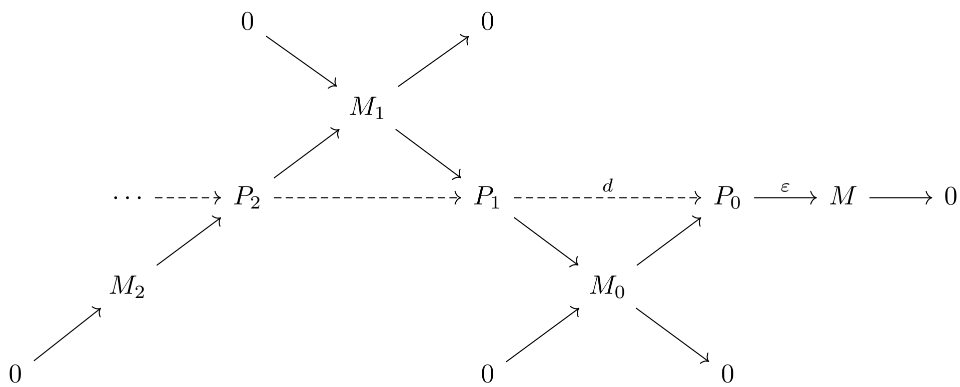
Definition 4.7. A projective resolution of an object $M \in \mathcal{A}$ is an exact sequence:

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

Definition 4.8. We say that a category $\mathcal{A}$ has enough projectives if for all $M \in \mathcal{A}$ there exists a projective P and an epi $P \to M \to 0$,

Proposition 4.9. *If an abelian category has enough projectives, then every object has a projective resolution.*

Proof. Wave your hands at the below diagram (or see Weibel Lemma 2.2.5).

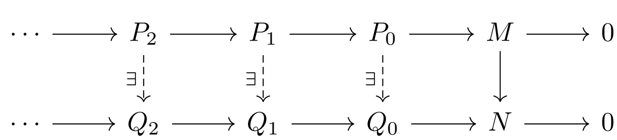


□

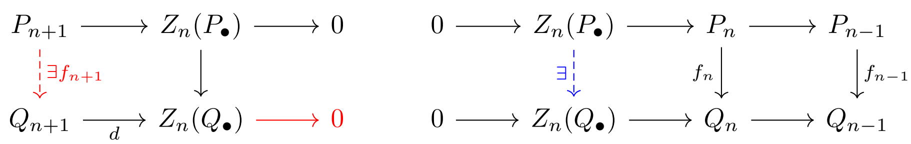
Now there are many possible projective resolutions of a given object, but it turns out they are related.

Theorem 4.10 (Comparison Theorem, Weibel Theorem 2.2.6)**.** *Let $P_\bullet \to M$ and $Q_\bullet \to N$ be projective resolutions and $f' : M \to N$ a morphism. Then there exists chain map $f : P_\bullet \to Q_\bullet$ lifting f'. Furthermore this lift is unique up to homotopy.*

Proof. We wish to show the existence of the following maps:



We can construct f_0 by projectivity. So assume by induction that f_i exists for all $i \leq n$, and consider "gluing together" the following diagrams (here we use exactness of $Q_\bullet$ and projectivity of P_{n+1}).



Finally we need to prove uniqueness up to homotopy. Suppose f and g are two lifts of f', and let $h := f - g$. We construct a null-homotopy of h by induction. Let $s_n = 0$ for $n < 0$. For $n = 0$ we have the following diagram gluing.

Now suppose by induction that s_i exists for $i < n$ such that $h = ds + sd$. Consider the map $h_n - s_{n-1}d : P_n \to Q_n$ and compute:

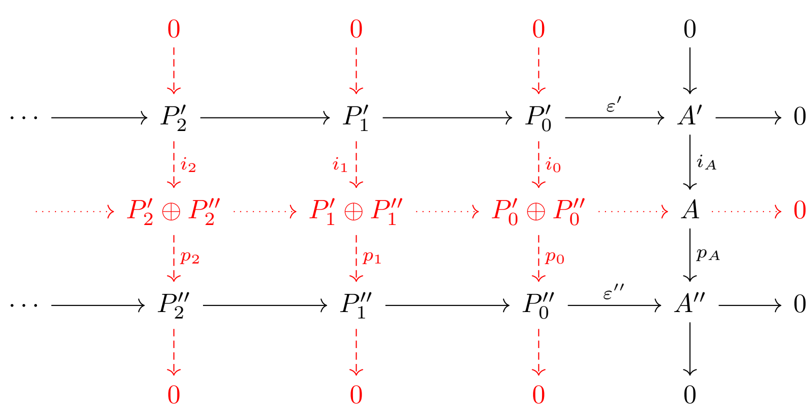
$$d(h - s_{n-1}d) = dh - (ds_{n-1})d = dh - (h - s_{n-2}d)d = dh - hd = 0$$

Therefore $h_n - s_{n-1}d$ lands in $Z_n(Q_\bullet)$, whence the following diagram:

gives an s_n such that $ds_n = h_n - s_{n-1}d$. □

Finally we come to a result regarding the interplay of projective resolutions and exact sequences.

Lemma 4.11 (Horeshoe Lemma)**.** Suppose we are given the following solid black horseshoe diagram, where the rows are projective resolutions and the column is exact.



We can fill in the diagram with the red to a projective resolution of A where i_n and p_n are the natural inclusions/projections.

Proof. We proceed by induction. Define ϵ to be the direct sum $i_A\epsilon' \oplus \tilde{\varepsilon}''$ where $\tilde{\varepsilon}'' : P_0'' \to A$ is a lift of ε''. Then the following diagram is commutative, with the exactness of the first column coming from the 9-lemma.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & \ker(\varepsilon') & \dashrightarrow & P_0' & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i_0} & & \downarrow & & \\
0 & \dashrightarrow & \ker(\varepsilon) & \dashrightarrow & P_0' \oplus P_0'' & \xrightarrow{\varepsilon = i_A\varepsilon' \oplus \tilde{\varepsilon}''} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p_0} & & \downarrow & & \\
0 & \dashrightarrow & \ker(e'') & \dashrightarrow & P_0'' & \xrightarrow{\varepsilon''} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We then draw the diagram, the red parts coming from exactness of the projective resolutions:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1' & \longrightarrow & \ker(\varepsilon') & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \ker(\varepsilon) & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & \ker(e'') & \longrightarrow & P_0'' & \longrightarrow & \ker(\varepsilon'') & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

And so the horseshoe is built by induction. ☐

4.2 Injective Objects

- definition
- divisibility and injectivity
- enough injectives in Ab and R-Mod.

4.2.1 Basic Definitions

Injective objects are categorically dual to projective objects in a precise way (see Remark 4.14).

Definition 4.12. An object $I \in \mathcal{A}$ is injective if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. Diagrammatically:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\forall i} & B \\ & & {\scriptstyle \forall f}\downarrow & \swarrow {\scriptstyle \exists \tilde{f}} & \\ & & I & & \end{array}$$

Definition 4.13. An abelian category has enough injectives if every object $M \in \mathcal{A}$ injects $0 \to M \to I$ into some injective object. An injective resolution is an exact sequence $(0 \to M \to I^0 \to I^1 \to \cdots)$, with I^n injective.

Remark 4.14. Since injective objects are projective objects of $\mathcal{A}^{\mathrm{op}}$ (!!), a category having enough injective implies the existence of injective resolutions. Furthermore the comparison and horseshoe lemma hold with arrows reversed.

4.2.2 Injectives in Ab

We begin by describing the injective objects in **Ab**.

Definition 4.15. An abelian group A is divisible if for all $n \neq 0$ and $a \in A$ there exists some $a' \in A$ such that $na' = a$.

Equivalently, $n \cdot : A \to A$ is surjective for all $n \neq 0$.

Examples 4.16. $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}[1/p]/\mathbb{Z}$ are all divisible, $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Z}[1/p]$ are not.

Proposition 4.17. *Divisible groups are injective.*

Proof. Let $A \subset B$ be injective and $f : A \to I$ a homomorphism. Consider the poset:

$$\mathcal{P} = \{(A', \alpha) : A \subset A' \subset B,\, \alpha : A' \to I, \alpha|_A = f\}$$

With $(A', \alpha') \leq (A'', \alpha'')$ iff $A' \subset A''$ and $\alpha''|_{A''} = \alpha'$. Now, ascending chains in $\mathcal{P}$ have upper bounds because the increasing union of subgroups is a subgroup. So by Zorn's lemma let (B', β) be a maximal element of $\mathcal{P}$.

If $B \neq B'$ choose $b \in B \setminus B'$, then either $\mathbb{Z}b \cap B' = 0$ or $nb\mathbb{Z}$ for some $n \neq 0$.

If $\mathbb{Z}b \cap B' = 0$, then $\mathbb{Z}b + B' \cong \mathbb{Z} \times B'$ and β extends over $\mathbb{Z}b + B'$ by the formula $\tilde{\beta}(nb, b') := \beta(b')$, contradicting maximality.

If $\mathbb{Z}b \cap B' = nb\mathbb{Z}$ then $\mathbb{Z}b + B' \cong \mathbb{Z} \times B'/(n, -nb)\mathbb{Z}$. Let $\beta(nb) = i$, then since I is divisible $i = ni'$ for some $i' \in I$. We can extend β by the formula:

$$\beta' : \mathbb{Z}b + B' \to I,\, rb + b' \mapsto ri' + \beta(b')$$

which is well-defined because of the above isomorphism.

Both cases contradict maximality, which means that $B' = B$, as required. $\square$

Fact 4.18. The categories **Ab** and $\mathbf{Mod}_R$ have enough injectives. See the exercise sheet or Weibel Pages 39 & 40

Fact 4.19. If $\mathcal{A}$ has enough injectives/projectives, then so does $\mathbf{Ch}(\mathcal{A})$.

Chapter 5

Lecture 5

5.1 δ-functors

- definition
- examples
- morphisms of δ-functors
- universality

5.1.1 Definition and Examples of δ-functors

We wish to formulate a universal property by which to eventually reason about derived functors.

Definition 5.1. A (covariant) homological δ-functor $T : \mathcal{A} \to \mathcal{B}$ between abelian categories, is the data of:

- a sequence of functors $T_n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$
- for every exact sequence $0 \to A \to B \to C \to 0$ a collection of morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$

such that:

1. For each SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ there is an LES:

   $$\cdots \xrightarrow{T_{n+1}(g)} T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \xrightarrow{T_n(f)} T_n(B) \xrightarrow{T_n(g)} T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \xrightarrow{T_{n-1}(g)} \cdots .$$

2. For each morphism of SESs

$$(0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0) \to (0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0)$$

a commutative diagram

$$\begin{array}{ccc} T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \\ \downarrow & & \downarrow \\ T_n(C') & \xrightarrow{\delta'} & T_{n-1}(A') \end{array}$$

Remark 5.2. To get the definition of a cohomological delta functor reverse the indices, to get the contravariant version swap A and C appropriately (see Weibel)

Examples 5.3.
- The prototypical example of a homological δ-functor is $(H_n) : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$.
- A trivial example is if T_0 is an exact functor and $T_n = 0 \, \forall \, n > 0$.

Exercise 5.4. Consider the functors $\mathbf{Ab} \to \mathbf{Ab}$, $T_0(A) := A/pA$ and $T_1(A) = \{a \in A : pa = 0\}$. Show that $(T_0, T_1, 0, ...)$ form a homological δ functor.

5.1.2 Morphisms of δ-functors

We just defined some objects, so we ought to define morphisms.

Definition 5.5. A morphism $\varphi : S_\bullet \to T_\bullet$ of δ-functors is a sequence of natural transformations $\varphi_n : S_n \to T_n$ commuting with all δ maps coming from short exact sequences.

Slogan 5.6. Essentially, for every SES there exists a commutative ladder diagram between the LESs associated to $S_\bullet$ and $T_\bullet$.

We are now in position to define a universal property.

Definition 5.7. A homological δ-functor T is universal if for any δ-functor S and natural transformation $f_0 : S_0 \to T_0$ there exists a unique morphism $\{f_n : S_n \to T_n\}$ extending f_0.

Remark 5.8. It is perhaps not obvious that the correct definition of a cohomological δ-functor goes in the other direction. I.e. T is universal if any natural transformation $f_0 : T_0 \to S_0$ extends uniquely to $f : T \to S$.

Example 5.9. Both δ-functors of Example 5.3 are universal, though H_* being universal will take some time to prove.

Question 5.10. If $T_0 := F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories, we can ask whether there is *any* δ-functor extending T_0. Certainly it is necessary, but insufficient for T_0 to be right exact. We will develop a sufficient condition in the next sections.

5.2 Derived functors

- construction & well-definedness.
- derived functors are δ-functors.
- derived functors are universal.

5.2.1 construction of derived functors.

We have now laid the groundwork to make the following central definition.

Construction 5.11. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, and suppose that $\mathcal{A}$ has enough projectives. We define the left derived functors of F, denoted $(L_i F)_{i \geq 0}$ by the following 3 step process.

Step 1: For every object $A \in \mathcal{A}$ *choose* a projective resolution:

$$P_\bullet \to A = (\cdots \to P_2 \to P_1 \to P_0 \to A \to 0)$$

Step 2: Apply F and drop A from the complex:

$$FP_\bullet = (\cdots \to FP_2 \to FP_1 \to FP_0 \to 0)$$

Step 3: Take homology:

$$L_i(A) := H_i(FP_\bullet) = \ker(FP_i \to FP_{i-1})/\mathrm{im}\,(FP_{i+1} \to FP_i)$$

Remark 5.12. It is left to the reader to define right derived functors using injective resolutions.

There are still things to check to justify calling these the left derived functors of F, we encapsulate these in the following theorem.

Theorem 5.13. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, and suppose that $\mathcal{A}$ has enough projectives, then:*

1. *Any map $f : A \to A'$ induces a map $L_i F(f) : L_i F(A) \to L_i F(A')$.*
2. *$L_i F$ are additive functors.*
3. *Different choices of projective resolutions yield naturally isomorphic functors. In particular, the isomorphism class of objects $L_i F(A)$ are independent of the choice in step 1.*

Proof. 1. Consider a morphism $f : A \to A'$, and projective resolutions of both objects. By the comparison lemma, there exists a chain map:

$$\begin{array}{ccc} P_\bullet & \longrightarrow & A \\ \downarrow{\scriptstyle \exists f_\bullet} & & \downarrow{\scriptstyle f} \\ P'_\bullet & \longrightarrow & A' \end{array}$$

This induces a unique map $H_i(FP_\bullet) \to H_i(FP'_\bullet)$, since $f_\bullet$ is unique up to homotopy.

2. To prove that L_i is functorial consider the following:

$$\begin{array}{ccc} P_\bullet & \longrightarrow & A \\ \downarrow{\scriptstyle f_\bullet} & & \downarrow{\scriptstyle f} \\ P'_\bullet & \longrightarrow & A' \\ \downarrow{\scriptstyle g_\bullet} & & \downarrow{\scriptstyle g} \\ P''_\bullet & \longrightarrow & A'' \end{array} \qquad (g\circ f)_\bullet : P_\bullet \dashrightarrow P''_\bullet$$

   Since both are lifts of $g \circ f$ we have a chain homotopy $(g \circ f)_\bullet \sim g_\bullet \circ f_\bullet$. Therefore $L_i F(g \circ f) = L_i F(g) \circ L_i F(f)$. Additivity is left to the reader.

3. Apply the argument of 2. with $A = A' = A''$ and $P''_\bullet = P_\bullet$ to get the maps forming a natural isomorphism.

□

Exercise 5.14. Prove that $L_0 F(A) \cong F(A)$ and $L_0 F(f) = F(f)$ under that identification.

5.2.2 Derived Functors as δ-Functors

One of the features of derived functors that makes them so powerful is the existence of long exact sequences associated to short exact sequences. Here we prove that derived functors for universal δ-functors.

Theorem 5.15 (Weibel Theorem 2.4.6)**.** *The derived functors $L_\bullet F$ form a homological δ-functor.*

Proof. We prove the existence of the δ maps, but omit the proof of naturality. Details can be found in Weibel.

Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence. Applying the Horseshoe lemma, we have projective resolutions:

$$\begin{array}{ccc} P'_\bullet & \longrightarrow & A' \\ \downarrow & & \downarrow \\ P_\bullet & \longrightarrow & A \\ \downarrow & & \downarrow \\ P''_\bullet & \longrightarrow & A'' \end{array}$$

with $P_n = P'_n \oplus P''_n$. Therefore since the sequence of complexes $P'_\bullet \to P_\bullet \to P''_\bullet$ is *split* exact, the maps of complexes $FP'_\bullet \to FP_\bullet \to FP''_\bullet$ is still exact, the corresponding LES of homology is:

$$\cdots \to L_n F(A) \to L_n F(A'') \xrightarrow{\delta} L_{n-1}(A') \to L_{n-1}(A) \to \cdots$$

providing us with the required δ maps. □

Theorem 5.16 (Weibel Theorem 2.4.7)**.** *The left derived functors L_*F of a right exact functor $F : \mathcal{A} \to \mathcal{B}$ form a universal δ-functor.*

Proof. We give only a proof indication. Suppose T_* is a homological δ-functor and $\phi_0 : T_0 \to L_0F$ is a natural transformation. We need to show that this extends to a unique map of δ-functors $\phi_* : T_* \to L_*F$.

We construct the natural transformation by induction. Suppose there exist already $\phi_i : T_i \to L_iF$ for $i < n$ commuting with all δ maps. For $A \in \mathcal{A}$, since $\mathcal{A}$ has enough projectives choose a SES $0 \to K \to P \to A \to 0$ with P projective. We have the following diagram:

$$\begin{array}{ccccccc}
 & & T_n(A) & \longrightarrow & T_{n-1}(K) & \longrightarrow & T_{n-1}(P) \\
 & & \Big\downarrow{\scriptstyle \phi_n} & & \Big\downarrow{\scriptstyle \phi_{n-1}} & & \Big\downarrow{\scriptstyle \phi_{n-1}} \\
L_nF(P) = 0 & \longrightarrow & L_nF(A) & \longrightarrow & L_{n-1}F(K) & \longrightarrow & L_{n-1}F(P)
\end{array}$$

The blue exists and is unique by a diagram chase.

One then checks that the ϕ_n do not depend on the choice of P, and then that they commute with all δ maps. □

5.3 Tor and Ext

- Tor and Ext definitions
- Balancing Tor and Ext.

We are now in a position to make the following definitions, which were the aim on the mini-course.

We have seen that the category $\mathbf{Mod}_R$ has both enough injective and projective objects, so the left/right derived functors of all right/left exact functors coming out of $\mathbf{Mod}_R$ exist. We are in good shape to make the following definitions.

Definition 5.17. Define $\mathrm{Tor}_i^R(A, B) := L_i(- \otimes B)(A) \cong L_i(A \otimes -)(B)$.

Definition 5.18. Define $\mathrm{Ext}_R^i(A, B) := R^i(\mathrm{Hom}_R(A, -))(B) \cong R_i(\mathrm{Hom}(-, B))(A)$.

Remark 5.19. Note that the contravariance of $\mathrm{Hom}_R(-, B)$ means that computing $R_i(\mathrm{Hom}_R(-, B))(A)$ requires taking a projective resolution of A.

Remark 5.20. There is a *BIG ISSUE* with the above definitions. We actually gave two different definitions, and claimed they were isomorphic. Let's remedy this situation.

Theorem 5.21 (Balancing Tor)**.** *For R-modules A and B, there is an isomorphism $L_i(- \otimes B)(A) \cong L_i(A \otimes -)(B)$.*

Proof. To compute the left hand side take a projective resolution $A \leftarrow P_\bullet$, and for the right hand side a projective resolution $B \leftarrow Q_\bullet$. We take the "tensor product" of these two resolutions to get a double complex:

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_2 \otimes B & \longleftarrow & P_2 \otimes Q_0 & \longleftarrow & P_2 \otimes Q_1 & \longleftarrow & P_2 \otimes Q_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_1 \otimes B & \longleftarrow & P_1 \otimes Q_0 & \longleftarrow & P_1 \otimes Q_1 & \longleftarrow & P_1 \otimes Q_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_0 \otimes B & \longleftarrow & P_0 \otimes Q_0 & \longleftarrow & P_0 \otimes Q_1 & \longleftarrow & P_0 \otimes Q_2 & \longleftarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & A \otimes Q_0 & \longleftarrow & A \otimes Q_1 & \longleftarrow & A \otimes Q_2 & \longleftarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Since the Q_i and P_i are projective, the rows and columns ending in 0 are all exact, since they are $P_i \otimes -$ applied to an exact sequence, and the P_i are projective, hence flat (Corollary 4.5).

One concludes by defining zig-zag maps $L_i(A \otimes -)(B) \leftrightarrow L_i(- \otimes B)(A)$ via diagram chase. Once one knows it is well-defined, we can check that the composition is the identity. Therefore we have constructed an isomorphism. □

Exercise 5.22. Prove the analogous result for Ext^*_R. Everything should work the same, just ensure that your arrows end up pointing in the right directions and that you know why the important columns/rows are exact.

Corollary 5.23 (Classification of flat modules)**.**
For an R-Module B, the following are equivalent:

1. $\operatorname{Tor}_i^R(A, B) = 0$ *for all* $A \in \mathbf{Mod}_R$ *and* $i > 0$.
2. $\operatorname{Tor}_1^R(A, B) = 0$ *for all* $A \in \mathbf{Mod}_R$.
3. $- \otimes B$ *is an exact functor. I.e. B is a flat R-module.*

Proof. Certainly 1. $\implies$ 2..

For 2. $\implies$ 3. If $0 \to A' \to A \to A'' \to 0$ is exact, then by the LES for Tor:

$$\cdots \to \underbrace{\operatorname{Tor}_1^R(A'', B)}_{=0} \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$$

Therefore $- \otimes B$ is exact.

For $3 \implies 1$ note that $(-\otimes B, 0, ...)$ and $L_*(-\otimes B)$ are both universal δ-functors with $T_0 = -\otimes B$. Therefore for $i > 0$:

$$\mathrm{Tor}_i^R(A, B) := L_i(-\otimes B)(A) \cong 0$$

$\square$