Homological Algebra Summer Mini-course 2024: Problem Sets

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0.1 Introduction

Theses are the lecture notes for the summer mini course on homological algebra I am running at UT Austin July 15th - 19th.

Contents

Problem set 0

This is problem set 0, intended as a warm-up for the summer mini-course course on homological algebra. Feel free to do as many or as few of them as you would like. Don't worry if you can't do them all right now.

Exercise 0.1. Understand why an exact sequence $0 \to A \to B \to C \to 0$ carries essentially the same data as an isomorphism $B/A \cong C$.

Exercise 0.2. If $f : A \to B$ is a map of modules, we define the $\text{coker}(f) := B/f(A)$. What is the kernel of the natural map $B \to \text{coker}(f)$?

Exercise 0.3. Let $\mathcal{C}$ be a category. An initial object $A \in \mathcal{C}$ is an object such that for all $B \in \mathcal{C}$ there exists a unique morphism $A \to B$. Prove that any two initial objects are uniquely isomorphic.

Hint: Suppose A and A' are both initial objects. Can you construct maps between them? what happens when you compose these maps?

Do the following categories have initial objects, and if so, what are they?: **Grp**, **Ring**, **Vect**, **Fields**, $\mathbf{Mod}_R$.

If this has you interested, define terminal objects and do the same exercise. An object that is both initial and terminal is called the *zero object* of $\mathcal{C}$.

Exercise 0.4.

- If $\{A_i\}_{i \in I}$ is a family of R-modules, we define the direct sum:

$$\bigoplus_{i \in I} A_i := \left\{ \sum_{i \in A_i} a_i \,:\, a_i \in A_i \text{ and } a_i = 0 \text{ for all but finitely many } i \in I \right\}$$

  Define for yourself the addition and R-multiplication and convince yourself that everything is well defined.

- Do the same for the product:

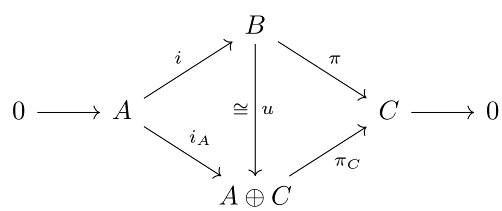
$$\prod_{i \in I} A_i := \{(a_i)_{i \in I} \,:\, a_i \in A_i\}$$

- Show that when I is a finite set that $\bigoplus_{i \in I} A_i \cong \prod_{i \in I} A_i$. How about when I is infinite?

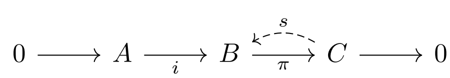
- Formulate/look up the universal property of both objects. This justifies the terminology "the" direct sum/product.

Exercise 0.5 (The splitting lemma; will be used without proof in lectures)**.** Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be an exact sequence of R-modules (take abelian groups if you prefer). Then prove (or look up a proof) that the following are equivalent.

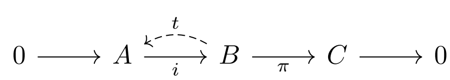
1. There exists an isomorphism $u : B \xrightarrow{\sim} A \oplus C$ fitting into a commutative diagram:



2. There exists a map $s : C \to B$ such that $\pi s = \mathrm{id}_C$.



3. There exists a map $t : B \to A$ such that $ti = \mathrm{id}_A$.



Hint: $1 \implies 2$ *and* 3 *should be straightforward. For* $2 \implies 1$ *use* $s\pi$ *to show that every element of* B *can be written as a sum of something in* $\ker(\pi)$ *and* $\mathrm{im}\,(s)$*. Use a similar strategy for* $3 \implies 1$.

Exercise 0.6. Let R be a commutative ring and let M be an R-module. Consider the mapping $N \mapsto M \otimes_R N$, which again gives us an R-module. Extend this mapping to morphisms $N \to N'$ to define a functor $M \otimes - : \mathbf{Mod}_R \to \mathbf{Mod}_R$.

Hint: the universal property of the tensor product will be useful. Don't forget to check that what you define is indeed a functor. I.e. respects composition.

Exercise 0.7. Let R be any ring, and let M be a left R-module and N an abelian group. Show that $\mathrm{Hom}_{\mathbf{Ab}}(M, N)$ is a right R-Module via the rule $(r \cdot f)(m) := f(rm)$.

Hint: If you like, forget about the left and right and assume that R is commutative.

Chapter 1

Problem set 1

1.1 Chain Complexes

- Chain complexes, homology
- Morphisms of chain complexes, quasi-isomorphisms.

Exercise 1.1 (Essential result, diagram chase).
Prove that a map of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ induces a map $f_* : H_*(C_\bullet) \to H_*(D_\bullet)$

Exercise 1.2 (Good practice for basic definitions).
Consider the following diagram:

$$\begin{array}{ccccccccc} \cdots 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots. \end{array}$$

Define maps d and f that make the rows chain complexes and f:

1. a quasi-isomorphism.
2. a chain map that is not a quasi-isomorphism.
3. not a map of chain complexes.
4. Prove that there is no map g in the other direction that is also a quasi-isomorphism.

Exercise 1.3 (Computation). Consider the sequence of abelian groups defined by: $C_n = 0$ for all $n < 0$, $C_n = \mathbb{Z}/8$ for all $n \geq 0$ and $d_n = \cdot 4$.

$$\cdots \xrightarrow{\cdot 4} \mathbb{Z}/8 \xrightarrow{\cdot 4} \mathbb{Z}/8 \xrightarrow{\cdot 4} \mathbb{Z}/8 \to 0$$

Verify this is a chain complex, and compute its homology groups.

Exercise 1.4 (I recommend skipping this now and coming back if you want more practice with diagram chasing).

Prove the 5-lemma. I.e., suppose you have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\big\downarrow\!\!\downarrow & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow & & \big\downarrow{\scriptstyle\cong} & & \big\hookdownarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Prove that the middle vertical map is an isomorphism.

1.2 Operations on Chain Complexes

- Shifting, direct sums, products, kernels, cokernels, exactness.

Exercise 1.5 (Definition practice, Weibel Exercise 1.2.1).

Prove that homology distributes over direct sums. Specifically:

$$H_n(C_\bullet \oplus D_\bullet) \cong H_n(C_\bullet) \oplus H_n(D_\bullet).$$

Hint: Unravel the definitions. The definition of the differential maps in $C_\bullet \oplus D_\bullet$ *are important.*

Exercise 1.6 (Weibel page 5, unimportant).

For a family of chain complexes $\{C_\lambda\}_{\lambda\in\Lambda}$ define the product of chain complexes $\prod_{i\in I} C_i$ and verify that $d^2 = 0$.

Hint: This can be done with only the universal property of the product, without writing down any elements. Of course feel free to write down elements if you wish.

Exercise 1.7 (Essential construction).

If $f_\bullet : B_\bullet \to C_\bullet$ is a map of complexes, check that the differentials for $\ker(f_\bullet)$ and $\operatorname{coker}(f_\bullet)$ are well-defined, and that $d^2 = 0$.

Hint: one approach is to extend the commutative diagram that is $f_\bullet$ *and diagram chase.*

Exercise 1.8 (Weibel exercise 1.2.7, good practice with SES of complexes).

If C is a chain complex, show that there are exact sequences of complexes:

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0$$

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0$$

Hint: some of the complexes here have not been explicitly defined. Most of this question is parsing what they ought to be, and then looking at the nth part.

1.3 Long Exact Sequence of Homology

- The Snake Lemma
- LES of Homology

Exercise 1.9 (Another diagram chase, used in the proof of LES of homology)**.**

Suppose you have the following diagram, with the solid rows and columns exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & Z_n(A) & \dashrightarrow & Z_n(B) & \dashrightarrow & Z_n(C) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_{n-1}/dA_n & \dashrightarrow & B_{n-1}/dB_n & \dashrightarrow & C_{n-1}/dC_n & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Prove the existence of the dashed lines and their exactness. (just prove a subset if you like).

Hint: You have done the existence part in Exercise 1.7

Exercise 1.10 (Diagram chase practice)**.**

In the set up of the snake lemma, check the exactness at as many spots in the claimed exact sequence:

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0$$

Exercise 1.11 (Simple consequence of the LES of homology, Weibel Exercise 1.3.1)**.**

Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a SES of chain complexes. Prove that if any two of them are acyclic (i.e. exact), then the third is too.

Exercise 1.12 (3x3 lemma/9-lemma, important, simple consequence of the LES of homology, Weibel Exercise 1.3.2)**.**

Consider the following commutative diagram of R-modules.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Suppose all three rows are exact, prove the following:

1. If the first two columns are exact, so is the last column.
2. If the last two columns are exact, so is the first column.
3. If the left and right columns are exact, and the middle column is a complex, then the middle column is exact.

Chapter 2

Problem set 2

2.1 Abelian Categories

- Ab $\subset$ additive categories.
- Monos, epis, kernels & cokernels.
- Examples in R-**Mod** and $\mathbf{Ch}(\mathcal{A})$.
- Abelian categories.
- Familiar notions in Abelian Categories.

Exercise 2.1 (Definition practice)**.**
Prove that in R-**Mod** , a morphism $f : A \to B$ is monic iff it is injective.
Hint: Consider $\{a \in A : f(a) = 0\} \to A$.

Exercise 2.2 (Important, Definition practice)**.**
Prove that kernels exist in R-**Mod** and coincide with the usual definition of kernel.
Hint: Copy the dual result proven in the notes.

Exercise 2.3 (Important fact to know, but you can skip it if other problems seem more interesting)**.**
If $f_\bullet : A_\bullet \to B_\bullet$ is a map of complexes, then $\ker(f_\bullet)$ and $\mathrm{coker}(f_\bullet)$ are kernels and cokernels respectively in the categorical sense.

As a consequence check that the definition of an exact sequence of chain complexes of R-Modules is not ambiguous with respect to the two a priori different definitions.

Hint: Just do one of these, look at the n*th component and use the universal property in* R-**Mod** *. Do you actually use the fact that it is* R-**Mod** *as opposed to an arbitrary abelian category anywhere?*

Exercise 2.4. There are a huge list of possible exercises I could put here that use the formalism of abelian categories, but I would suggest not getting bogged down and moving on to the more interesting stuff. If you are really desperate to get practice with the formalism, take any fact about abelian groups, then translate and prove it for abelian categories. E.g. kernels are monic, or if g is monic, then $\ker(g \circ f) = \ker(f)$.

2.2 Chain Homotopies

- Chain homotopies & induced map on homology
- Split complexes

Exercise 2.5 (The homotopy Category (Weibel Ex.1.4.5))**.**

The following exercise gives an interesting example of a category which is additive but not abelian. The category is often called the homotopy category. **By far the most interesting part of this exercise is part (d).**

(a) We start with the category $\mathbf{Ch}(\mathbf{Ab})$ of chain complexes of abelian groups. Prove that homotopy equivalence is an equivalence relation on $\mathrm{Hom}_{\mathbf{Ch}(\mathbf{Ab})}(C, D)$.

For chain complexes C and D, define $\mathrm{Hom}_{\mathbf{K}}(C, D) := \mathrm{Hom}_{\mathbf{Ch}(\mathbf{Ab})}(C, D)/\sim$, where $f \sim g$ is homotopy equivalence. Show that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group.

(b) Suppose $f \sim g : C \to D$ and that $u : B \to C$ and $v : D \to E$, then $vfu \sim vgu$. Conclude that there is a category $\mathbf{K}$ with objects chain complexes and morphisms as described in (a). (i.e. Prove that composition is well-defined, and hence associative.)

(c) If $f_1, f_2, g_1, g_2 : C \to D$ are chain maps such that $f_i \sim g_i$, show that $f_1 + f_2 \sim g_1 + g_2$. Conclude that $\mathbf{K}$ is an additive category, and that $\mathbf{Ch}(\mathbf{Ab}) \to \mathbf{K}$ is an additive functor.

(d) Prove that $\mathbf{K}$ is not an abelian category by proving that:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Has no kernel.

Hint: Here is one approach: let $B_\bullet$ be the kernel, use the universal property to show it is concentrated in degree 0. Chain complexes with a singular copy of $\mathbb{Z}$ could be useful.

Prove that B_0 is the kernel of $\mathbb{Z} \to \mathbb{Z}/2$ directly (reduce to $\mathbf{Ab}$).

Prove that B can't be the kernel using the complex:

$$\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \to \cdots$$

Exercise 2.6 (Easy, good practice, inessential)**.**

Let $C_\bullet$ be a chain complex and $s_n : C_n \to C_{n+1}$ **randomly chosen maps**. Define $f : C_\bullet \to C_\bullet$ by $f := ds + sd$. Prove that f is a map of chain complexes.

2.2.1 Split Complexes

These two exercise are inessential, but serve as good practice.

Exercise 2.7 (example of a non-split complex, definition practice)**.** Consider the complex of abelian groups:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \cdots$$

Prove that $C_\bullet$ is exact, but not split.

Exercise 2.8 (Justifies the definition of a split complex)**.**

Prove that an exact sequence is split if and only if there are decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$. Furthermore $C_\bullet$ is split exact if and only if $H'_n = 0$

Hint: There are short exact sequences associated to $Z_n \subset C_n$ and $B_n \subset Z_n$. Use the splitting lemma for R-Modules and $dsd = d$ to show that these exact sequences split.

Chapter 3

Problem set 3

3.1 Fundamental Results on Abelian Categories

- Additive functors and exactness.
- Freyd-Mitchell Embedding Theorem and consequences.
- Left exactness of Hom
- Yoneda for additive functors.

Exercise 3.1 (Left exactness of Hom, essential result)**.**
Let $M \in \mathcal{A}$ be an object in an abelian category, prove that $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathbf{Ab}$ is a (covariant) left-exact functor.

Exercise 3.2 (Left exactness of contravariant Hom, essential result)**.**
Let $M \in \mathcal{A}$ be an object in an abelian category, prove that $\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A} \to \mathbf{Ab}$ is a contravariant left-exact functor.
Hint: If $\mathcal{A}$ is an abelian category, then so is $\mathcal{A}^{\mathrm{op}}$.

Exercise 3.3 (The contravariant Yoneda lemma for abelian categories, good practice, useful result.)**.**
Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of morphisms in an abelian category $\mathcal{A}$. If for all objects $M \in \mathcal{A}$ the sequence:

$$\mathrm{Hom}_{\mathcal{A}}(C, M) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(B, M) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A, M)$$

is exact, then the original sequence is exact.
Hint: Mimic the proof for the covariant case, or perhaps prove it via sleight of hand using $\mathcal{A}^{\mathrm{op}}$. Can you prove it without using Freyd Mitchell?

Exercise 3.4 (Limitations of Freyd-Mitchell)**.**
What is wrong with the following "solution" to the covariant version of the above exercise?

Proof. By Freyd-Mitchell we may assume for a fixed sequence that $\mathcal{A}$ is $\mathbf{Mod}_R$. Then setting $M = R$ we have natural isomorphisms $\mathrm{Hom}_{\mathbf{Mod}_R}(R, A) \cong A$ yielding a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{Mod}_R}(R, A) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathbf{Mod}_R}(R, B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathbf{Mod}_R}(R, C) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad g \quad} & C
\end{array}$$

whence the result. □

3.2 Adjoints

The amount of things to check in these exercises can be overwhelming. It might help to start with Exercise 3.6.

Exercise 3.5 (Tensor-Hom adjunction, essential)**.**
Let B be an R-S bi-module and consider the morphisms:

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)), \tau f(a) : b \mapsto f(a \otimes b)$$

One can also describe τ by $f \mapsto (a \mapsto (b \mapsto f(a \otimes b)))$.

Convince yourself that the following dot points are everything that needs to be proven to show the Tensor-Hom adjunction, then prove the least obvious one for yourself.

- $\tau f(a)$ is an S-module map for all $a \in A$.
- τf is a map of R-modules.
- τ is a map of abelian groups.
- ηg is a map of S-modules.
- τ has inverse given by $\eta : g \mapsto (a \otimes b \mapsto g(a)(b))$.
- τ is natural in C.
- τ is natural in A.

Exercise 3.6 (A right adjoint to forgetful)**.**
Consider $\mathrm{Hom}_{\mathbf{Ab}}(R, -) : \mathbf{Ab} \to \mathbf{Mod}_R$, and let $(-)_{\mathbb{Z}} : \mathbf{Mod}_R \to \mathbf{Ab}$ be the forgetful functor. Check that $(-)_{\mathbb{Z}} \dashv \mathrm{Hom}_{\mathbf{Ab}}(R, -)$ via τ, where $M \in \mathbf{Mod}_R$ and $A \in \mathbf{Ab}$:

$$\tau : \mathrm{Hom}_{\mathbf{Ab}}(M_{\mathbb{Z}}, A) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Ab}}(R, A)), (\tau f)(m) : r \mapsto f(rm)$$

One can also describe τ by $f \mapsto (m \mapsto (r \mapsto f(mr)))$.

Hint: If you did Exercise 3.5 first you can recover this with a clever choice S and B

Exercise 3.7 (A left adjoint to forgetful)**.**

Let $\phi : S \to R$ be a map of commutative rings. The functor $- \otimes_S R : \mathbf{Mod}_S \to \mathbf{Mod}_R$ is left adjoint to $(-)_S : \mathbf{Mod}_R \to \mathbf{Mod}_S$. Given an R-module M, M_S is simply M viewed as an S-module via $\phi : S \to R$. Write down the counit/unit isomorphisms for this adjunction.

Conclude that $\mathrm{Hom}_{\mathbf{Ab}}(R, -) : \mathbf{Ab} \to \mathbf{Mod}_R$ is right adjoint to an exact functor. What does this tell you about the exactness of $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$?

Hint: I believe this is easiest to just do by hand, but it might be possible to conclude it as a consequence of Tensor-Hom as well.

Chapter 4

Problem set 4

4.1 Injective Objects

- Computations.
- **Ab** and $\mathbf{Mod}_R$ have enough injective objects.

4.1.1 Computations

Exercise 4.1 (Computational)**.**
Working in **Ab**, find injective resolutions of the following groups: $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}/n, \mathbb{R}, \mathbb{R}^*, \mathbb{C}, \mathbb{C}^*$

Exercise 4.2 (Good practice, inessential)**.**
Prove that injective abelian groups are divisible.
Hint: This is the converse of the theorem proved in class. Cook up a monomorphism to apply injectivity to.

4.1.2 Injective objects in familiar categories

The folowing suite of exercises walks you through the proof that **Ab** and $\mathbf{Mod}_R$ have enough injective objects.

Exercise 4.3 (Essential, **Ab** has enough injectives)**.**
Let A be an abelian group, and consider $I(A) := \prod_{\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$. There is a natural map:

$$e_A : A \to I(A),\ e_A(a) := (\phi(a))_{\phi \in \mathrm{Hom}(A,\mathbb{Q}/ZZ)}$$

Prove that $I(A)$ and e_A are injective (module and homomorphism respectively).
This shows that **Ab** has enough injectives.

Exercise 4.4 (Essential, $\mathbf{Mod}_R$ has enough injectives)**.**

(a) Show that if an abelian group $A \neq 0$, then $\mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$.
Hint: Remember that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group

(b) Now let M be an R-module and use the fact that $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is an R-module to construct a similar map:

$$e_M : M \to I(M) = \prod_{\eta \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$$

and prove that e_M is injective.

Hint: First show that there is a non-zero η, the adjunction from Exercise 3.6 $(-)_{\mathbb{Z}} \dashv \mathrm{Hom}_{\mathbf{Ab}}(R, -)$ *might help.*

(c) Prove that $\mathrm{Hom}_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z})$ is an injective R-module.

Hint: Recall again that $(-)_{\mathbb{Z}} \dashv \mathrm{Hom}_{\mathbf{Ab}}(R, -)$*, then prove that a functor that is right adjoint to an exact functor preserves injective objects.*

(d) Conclude that $\mathbf{Mod}_R$ has enough injectives.

4.2 Projective Objects

- Projectives in $\mathbf{Mod}_R$.
- Projective resolutions.
- Comparison theorem.
- Horseshoe lemma.

4.2.1 Computations

Exercise 4.5 (Computations, useful for later)**.**
Working in **Ab**, find projective resolutions of: $\mathbb{Z}$, 0, $\mathbb{Z}/n$, $\mathbb{Z} \times \mathbb{Z}/n$ and then any finitely generated abelian group.

Exercise 4.6 (Computations)**.**
Let $R = \mathbb{Z}/12$, find projective resolutions in $\mathbf{Mod}_R$ of:
$\mathbb{Z}/12$, $\mathbb{Z}/6$ $\mathbb{Z}/4$ & $\mathbb{Z}/2$.

Exercise 4.7 (Computations)**.**
Let k be a field and let $R = k[t]/(t^2)$, find projective resolutions in $\mathbf{Mod}_R$ of $k[t]/(t^2)$ and k.

Exercise 4.8 (Computations, useful for later)**.**
Let k be a field and let $R = k[t]$, find projective resolutions in $\mathbf{Mod}_R$ of $k[t]/(t-a)$ and $k[t]/(t)$.

4.2.2 Projective objects in general

Exercise 4.9 (Quick)**.**

Prove that $P \in \mathcal{A}$ is projective (resp. injective) if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-, P)$) is an exact functor.

Hint: This is just a combination of a result from lectures and a reinterpretation of the definition of projective/injective.

Exercise 4.10 (Adjoints and projectives, Weibel Proposition 2.3.10)**.**

Prove that if $L : \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor, then L sends projective objects of $\mathcal{A}$ to projective objects of $\mathcal{B}$.

Prove the dual statement for a right adjoint to preserve injective objects.

Hint: Use the previous exercise and the Yoneda lemma for abelian categories.

Exercise 4.11 (Short and sweet, Weibel 2.2.3)**.**

Show that if $P_\bullet$ is a complex of projectives, with $P_i = 0$ for $i < 0$, then a map $\varepsilon : P_0 \to M$ giving a resolution of M is the same as a quasi-isomorphism $P_\bullet \to M$, where M is regarded as a complex concentrated in degree 0.

Exercise 4.12 (Limitations of Freyd-Mitchell)**.**

If $\mathbf{Mod}_R$ has enough projectives, does that imply that any abelian category $\mathcal{A}$ has enough projectives? What if $\mathcal{A}$ is small?

Exercise 4.13 (Understanding the Horseshoe lemma, not used later)**.**

In the setting of the horseshoe lemma, prove that the constructed maps $d : P'_n \oplus P''_n \to P'_{n-1} \oplus P''_{n-1}$ are "upper triangular". Can you figure out what the diagonal entries are?

Hint: If a map $f : A \oplus A' \to B \oplus B'$ is given by a matrix, are there functions I can compose f with to isolate the entries of this matrix?

Exercise 4.14 (Open ended)**.**

If there was a result about projective objects that you didn't quite follow, then prove the analogous result for injective objects by mimicking the proof for projectives.

Chapter 5

Problem set 5

5.1 δ-functors

- definition
- examples
- morphisms of δ-functors
- universality

Exercise 5.1 (Hands-on example, good definition practice)**.**
Consider the functors $\mathbf{Ab} \to \mathbf{Ab}$, $T_0(A) := A/pA$ and $T_1(A) = \{a \in A : pa = 0\}$. Show that $(T_0, T_1, 0, ...)$ form a homological δ functor.
Hint: To construct δ apply the snake lemma to multiplication by p. This is actually $\mathrm{Tor}_*(\mathbb{Z}/p\mathbb{Z}, -)$, can you see why?

Exercise 5.2 (Trivial example, surprisingly used later)**.**
If T_0 is an exact functor and $T_n = 0 \, \forall \, n > 0$. Prove that (T_n) form a universal δ-functor.

5.2 Derived Functors

- construction & well-definedness.
- derived functors are δ-functors.
- derived functors are universal.

5.2.1 Basic Facts

Exercise 5.3 (Important fact)**.**
Prove that $L_0F(A) \cong F(A)$ and $L_0F(f) = F(f)$ under that identification. We often simply write $L_0F = F$.
Hint: Open up the 3 step construction and use the left exactness of F.

Exercise 5.4 (Quick and easy).

Prove that if A is projective then $L_iF(A) = 0$ for $i > 0$.

Exercise 5.5 (Good practice with the construction of derived functors).

Let $F : \mathcal{A} \to \mathcal{B}$ have left derived functors, and $U : \mathcal{B} \to \mathcal{C}$ be an exact functor. Prove that:

$$U(L_iF(A)) \cong L_i(UF)(A)$$

As a corollary, since the forgetful functor $\mathbf{Mod}_R \to \mathbf{Ab}$ is exact, one can forget about the R-Module structure when computing Derived functors into $\mathbf{Mod}_R$.

Hint: Open up the 3 step construction and see what's going on.

5.2.2 Acyclic Resolutions

Exercise 5.6 (Optional, good practice.).

If $0 \to M \to P \to A \to 0$ is exact with P projective, show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$, and that $L_1F(A) \cong \ker(FM \to FP)$

Exercise 5.7 (Optional, Weibel Exercise 2.4.3).

Show that if $0 \to M \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$ is exact with P_i all projective, then $L_iF(M) \cong L_{i-m-1}F(A)$ for all $i \geq m+2$, and that $L_{m+1}(M) \cong \ker(FM \to FP_m)$.

Exercise 5.8 (Optional, acyclic resolutions, Weibel Lemma 3.2.8). Let F be a right exact functor whose left derived functors exist. Let $Q_\bullet \to A$ to an acyclic resolution of A. That is, a resolution, with the Q_j all F-acyclic. Prove that $H_n(FQ_\bullet) \cong L_n(A)$ for all n.

Hint: For $n = 0$, apply the method of Exercise 5.3. For $n = 1$ truncate the resolution to get a SES, then apply Exercise 5.6. For $n > 1$ use the SES of $n = 1$ and induction. See Weibel for details.

5.3 Balancing $\mathrm{Ext}^*_R(A, B)$

Exercise 5.9 (Balancing Ext).

Construct an isomorphism $R\mathrm{Hom}(A, -)(B) \cong R\mathrm{Hom}(-, B)(A)$ following the proof method for Tor.

Hint: Everything should work the same as with Tor*, just ensure that your arrows end up pointing in the right directions and that you know why the important columns/rows are exact*

Exercise 5.10 (Classification of Projective modules).

Prove that the following are equivalent:

1. B is an injective R-module.
2. $\mathrm{Hom}_R(-, B)$ is an exact functor.
3. $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \neq 0$ and $A \in \mathbf{Mod}_R$.

4. $\mathrm{Ext}^1_R(A, B) = 0$ for all $A \in \mathbf{Mod}_R$

Hint: 1 $\iff$ 2 is an earlier exercise. For 2 $\implies$ 3 try to leverage the theory built in lecture. For extra difficulty, use only the definition of $\mathrm{Ext}(A, B) := R\mathrm{Hom}(A, -)(B)$

5.4 Computation Practice

Exercise 5.11 (Computations of Ext for abelian groups)**.**
Use both definitions of Ext to check the following table. Values of A go across, values of B go down:

$\mathrm{Ext}^1(A, B)$	$\mathbb{Z}$	$\mathbb{Z}/n$
$\mathbb{Z}$	0	$\mathbb{Z}/n\mathbb{Z}$
$\mathbb{Z}/m$	0	$\mathbb{Z}/\gcd(n, m)\mathbb{Z}$

Exercise 5.12 (Computations of Tor for abelian groups)**.**

(a) Prove that $\mathrm{Tor}_0(\mathbb{Z}, A) = A$ and $\mathrm{Tor}_i(\mathbb{Z}, A) = 0$ if $i \geq 1$.

(b) Prove that $\mathrm{Tor}_0(\mathbb{Z}/n, A) = A/nA$ and $\mathrm{Tor}_1(\mathbb{Z}/n, A) = \{a \in A : na = 0\}$

Exercise 5.13. Let $R = \mathbb{Z}/4$, work in $\mathbf{Mod}_R$, compute $\mathrm{Tor}^{\mathbb{Z}/4}_*(A, B)$ for $A, B \in \{\mathbb{Z}/4, \mathbb{Z}/2\}$. You should get different groups to the previous exercise.

Exercise 5.14 (Computation of Tor for ideals)**.**
Let $I \subset R$ be an ideal.
Prove that $\mathrm{Tor}_{i+1}(I, M) \cong \mathrm{Tor}_i(R/I, M)$ for $i > 0$ and that $\mathrm{Tor}_1(I, M) = \ker(I \otimes_R M \to M)$.
Hint: Use the exact sequence $0 \to I \to R \to R/I \to 0$

Exercise 5.15. Let $R = k[t]$, $A = k[t]/(t-a)$ and $B = k[t]/(t-b)$ with $a, b \in k$. Compute $\mathrm{Tor}_*(A, B)$ when $a \neq b$ and when $a = b$. What is:

$$\sum_{i \geq 0} (-1)^i \dim_k \mathrm{Tor}_i(A, B)$$

Fun fact: This alternating sum of dimension of Tor is one way to define intersection numbers on algebraic varieties. Can you see what the computations you just made are indicating geometrically? The fact that the higher Tor modules are not captured in the rings themselves is one of the motivations for derived algebraic geometry.