THE ARF AND ROHKLIN'S INVARIANT

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Abstract. These are notes to be presented at a summer mini-course at UT-Austin. These notes are not original, all mistakes are to be attributed to the author, and all credit goes to the texts: An introduction to Knot Theory by W.B. Raymond Lickorish, Lectures on the Topology of 3-Manifolds by Nikolai Saveliev, Notes by Anthony Conway, and Alexandra Kjuchukova.

Contents

These course notes' main objective is to introduce a classical invariant of quadratic forms on Z/2-vector space, and its many applications. We will assume familiarity with 4-manifolds, and knot theory, but these notes should be largely self-contained. We will start with a short review of Seifert Forms, Seifert Matrices, and S-equivalence. Then we will introduce quadratic forms over Z/2 and provide a complete invariant of these quadratic forms (The Arf-invariant). Next, we will see how we can use quadratic refinements of Seifert Matrices to give us an Arf-Invariant for knots. Furthermore, we will connect the Arf-Invariant to 4-dimensional topology first by proving the Arf-invariant obstructs knots being slice , then using the Arf-Invariant to prove a famous restriction of the signature of Spin 4-manifolds (Rokhlin's Theorem). Lastly, we will provide an invariant for Integer Homology 3-spheres, and highlight recent research using the Arf-invariant.

1. Review of Seifert Matrices

1.1. Presentation Matrices.

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Definition 1.1. Let M is a module over a commutative ring R. M is 2 *free* if any element in M can be uniquely expressed as a linear sum of elements in a base. A *finite presentation* for M is an exact sequence

$$F \xrightarrow{\alpha} E \xrightarrow{\phi} M \to 0$$

where E and F are free $R-$modules with finite bases. If α is represented by matrix A with respect to bases $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ of E and F, then the matrix A, of m rows and n columns, is a presentation matrix for M. Since ϕ is a surjection, the images of (e_m) can be thought of as generators for M, and the images of f_n as relations amongst those generators.

Theorem 1.2. *Any two presentation matrices A_0 and A_1 for M differ by a sequence of matrix moves of the following forms and their inverses:*

i Perutation of rows or columns;

ii Replacement of the matrix A_0 by $A_1 \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$;

iii Addition of an extra column of zeros to the matrix A_0;

iv Addition of a scalar multiple of a row (or column) to another row (or column).

1.2. **Construction Modulo Proofs.**

Proposition 1.3. *Suppose that F is a connected, compact, orientable surface with non-empty boundary, locally flat in S^3. Then the homology groups $H_1(S^3 - F; \mathbb{Z})$ and $H_1(F; \mathbb{Z})$ are isomorphic, and there is a unique non-singular bilinear form*

$$\beta : H_1(S^3 - F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$$

with the property that $\beta([c], [d]) = lk(c, d)$ for any oriented simple closed curves c and d in $S^3 - F$ and F respectively.

Proof outline. The isomorphism between $H_1(S^3 - F; \mathbb{Z})$ and $H_1(F; \mathbb{Z})$ given by Alexander Duality and Poincare Duality. For a given surface in S^3, F one can show $H_1(F; \mathbb{Z})$ is isomorphic to $\bigoplus_{2g+n-1} \mathbb{Z}$ and we can realize the generators of this group as a set of simple closed curves on F, $\{[f_i]\}$. Now consider a regular neighborhood of F called V. $F \hookrightarrow V$ is a homotopy equivalence and $\partial V = F \sqcup -F$. The inclusion of $\partial V \hookrightarrow V$ will induce a map on homology mapping one set of generators F to F and $-F$ to zero. Furthermore, the orientations of $[e_i]$ half the generators of $H_1(\partial V; \mathbb{Z})$ can be chosen so that $lk(e_i, f_j) = \delta_{ij}$. Lastly we note that by a Mayer-Vietoris argument $H_1(S^3 - F) \cong H_1(S^3 - V)$ hence generated by the curves $\{[e_i]\}$
Now define

$$\beta : H_1(S^3 - F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$$

by $\beta([e_i], [f_j]) = \delta_{ij}$. Now suppose $[c] \in S^3 - F, [d] \in F$ are any oriented simple closed curves. You can represent both as linear combinations of generating elements and then note $lk(c, f_j) = [c] = \lambda_i e_i = \lambda_i$. Consider this similarly for $[d]$, and we arrive at $\beta([c], [d]) = lk(c, d)$. □

Definition 1.4. Associated to the Seifert surface F for an oriented link L is the Seifert form

$$\alpha : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$$

defined by $\alpha(x, y) = \beta((i^-)_* x, y)$. $i^\pm$ is defined as the image of $F \subset S^3 - (F \times [-1, 1])$ under the inclusion map. The positive and negative signs are to specify if F is included to the top or bottom of $F \times [-1, 1]$, where $S^3 - (F \times [-1, 1]) \simeq S^3 - F$.

Taking a basis $\{[f_i]\}$ for $H_1(F;\mathbb{Z})$ with a dual β basis $\{[e_i]\}$ for $H_1(S^3 - F;\mathbb{Z})$ as before, α is represented by the Seifert matrix A, where

$$A_{ij} = \alpha([f_i],[f_j]) = lk(f_i^-, f_j) = lk(f_i, f_j^+)$$

An immediate consequence is that $H_1(S^3 - F;\mathbb{Z})$, $[f_i^-] = \sum_i A_{ij}[e_i]$, and $[f_j^+] = \sum_j A_{ij}[e_j]$.

Theorem 1.5. *Suppose that F_1 and F_2 are Seifert surfaces for an oriented link L in S^3. Then there exists a sequence of Seifert surfaces $\Sigma_1, \Sigma_2, \cdots, \Sigma_N$, with $\Sigma_1 = F_1$, $\Sigma_N = F_2$, such that for each i, either Σ_i is obtained from Σ_{i-1} or Σ_{i-1} is obtained from Σ_i by surgery along an arc embedded in S^3, or related by isotopy.*

Proof. Lickorish Chapter 8 □

Definition 1.6. Let A be a square matrix over $\mathbb{Z}$. An elementary enlargment of A is a matrix B of the form

$$B = \begin{bmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} A & 0 & 0 \\ \eta^\tau & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

for some column ξ or row η^τ. The matrix A is called the elementary reduction of B.

Definition 1.7. Let A and B be square matrices. A is unimodular congruent to B if there exists a square matrix P such that $B = P^\tau AP$, where $detP = \pm 1$.

Definition 1.8. Square matricies A and B over $\mathbb{Z}$ are called $S-$equivalent if they are related by a sequence of elementary enlargements, elementary reductions and unimodular congruences.

Theorem 1.9. *Let A and B be Seifert matrices for an oriented link L. Then A and B are $S-$equivalent.*

2. Arf Invariant

2.1. **The Arf-invarint of a quadratic form.** Let V be a finite dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.1. A function $q : V \to \mathbb{Z}/2\mathbb{Z}$ is called a quadratic form if there exists a associated bilinear form $I(x,y) = q(x+y) - q(x) - q(y)$ over $\mathbb{Z}/2\mathbb{Z}$.

Remark 2.2. Note that I is symmetric in that $I(x,y) = I(y,x)$, and alternating $I(x,x) = q(2x) - 2q(x) = 0$, and $q(0) = 0$.

Definition 2.3. A quadratic form q is called non-degenerate if its bilinear form I is non-degenerate.

Definition 2.4. Let $\langle -,- \rangle : V \otimes V \to k$ be a bilinear form (k in general is a ring, but is typically taken to be a field). A quadratic function $q : V \to k$ is called a **quadratic refinement** of $\langle -,- \rangle$ if

$$\langle -,- \rangle = q(v+w) - q(v) - q(w) + q(0)$$

for all $v, w \in V$. If such a q is indeed a *quadratic form* in that $q(tv) = t^2 q(v)$ then $q(0) = 0$ and $\langle v,v \rangle = 2q(v)$. This means that a quadratic refinement by a quadratic form always exists when $2 \in k$ is invertible.

We will later see that one way to express quadratic refinements is by characteristic elements of a bilinear form. Now let us move over to an example:

Example 2.5. Let $U = (\mathbb{Z}/2\mathbb{Z})^2$ have a basis a, b. There is only one non-degenerate symmetric bilinear form I on U given by $I(a,a) = I(b,b) = 0$ and $I(a,b) = 1$. Define the quadratic forms $q_0, q_1 : U \to \mathbb{Z}/2\mathbb{Z}$ by the formulas $q_0(a) = q_0(b) = 0,\ q_0(a+b) = 1,\ q_1(a) = q_1(b) = q_1(a+b) = 1$. The associated bilinear forms of both q_0 and q_1 equal the form I. However, the quadratic forms q_0 and q_1 are not equivalent since the form q_0 sends a majority of vectors of U to 0 while q_1 sends a majority of vectors of U to 1.

Claim 2.6. It turns out that any other any other non-degenerate quadratic form q on U is equivalent to either q_1 or q_0.

Proof of claim. Consider the form q with $q(a) = 0$ and $q(b) = 1$. We preform a change of basis to $a' = a,\ b' = a + b$ to get $q(a') = 0$ and $q(b') = q(a+b) = I(a,b) + q(a) + q(b) = 0$. Thus q is equivalent to q_0. □

Definition 2.7. A bilinear form is symplectic if it is bilinear, alternating, and non-degenerate. A given basis $\mathbf{e}_i, \mathbf{f}_i$ for a bilinear form ω is called a symplectic basis if $\omega(\mathbf{e}_i, \mathbf{e}_j) = 0 = \omega(\mathbf{f}_i, \mathbf{f}_j),\ \omega(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij}$.

Lemma 2.8. *For any non-degenerate quadratic form $q : V \to \mathbb{Z}/2\mathbb{Z}$, there exists a symplectic basis. In particular, $\dim V$ is even.*

Proof. Choose a basis in V, then the form $I(-,-)$ is given by a matrix I with $detI = 1$, and $I(x,y) = x \cdot Iy$. If $x \neq 0$ there exists u such that $x \cdot u = 1$, and hence $I(x,y) = 1$ for $y = I^{-1}u$. The vectors x and y are linearly independent because $I(x,y) = 1$; in particular, $dimV \geq 2$.
Choose a new basis in V with the first two vectors x and y. The matrix I in this new basis takes the form

$$\begin{bmatrix} H & * \\ * & I_0 \end{bmatrix} \text{ where } H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By elementary transformation it can be turned into $\begin{bmatrix} H & * \\ * & I_1 \end{bmatrix}$, and induction completes the proof. □

Definition 2.9. Let $q : V \to \mathbb{Z}/2\mathbb{Z}$ be a non-degenerate quadratic form, and $a_i, b_i,\ i = 1, \cdots, n$, a symplectic basis in V. Define the **Arf-invariant** of q by the formula

$$\text{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}/2\mathbb{Z}$$

Now we need show that $\text{Arf}(q)$ is independent of the choice of a symplectic basis. This will follow from the upcoming study of the non-degenerate quadratic forms over $\mathbb{Z}/2\mathbb{Z}$.

Example 2.10. The forms $q_0, q_1 : U \to \mathbb{Z}/2\mathbb{Z}$ from the example above have Arf-invariants $\text{Arf}(q_0) = 0$ and $\text{Arf}(q_1) = 1$. Thus, the Arf-invariant provides a complete classification of non-degenerate quadratic forms on U. We will prove that the Arf-invariant completely classifies quadratic forms in general.

Lemma 2.11. *On $U \oplus U$, the forms $q_0 + q_0$ and $q_1 + q_1$ are equivalent.*

Proof. Note that q_0+q_0 and q_1+q_1 have the same associated bilinear form on $U\oplus U$. Let $a_j, b_j,\ j=1,2$, be a basis for $U\oplus U$ so that a_j, b_j form a symplectic basis on the j^{th} copy of U. If $\psi_i = q_i + q_i$ for $i=0,1$, then $\psi_0(a_j)=\psi_0(b_j)=0$ and $\psi_1(a_j)=\psi_1(b_j)=1,\ j=1,2$. Choose a new basis for $U\oplus U$,

$$a_1' = a_1+a_2,\ b_1' = b_1+a_2,\ a_2' = a_2+b_2+a_1+b_1,\ b_2' = b_2+a_1+b_1.$$

This defines a symplectic basis and $\psi_1(a_j')=\psi_0(a_j)$ and $\psi_1(b_j')=\psi_0(b_j),\ j=1,2$ so that ψ_1 is equivalent to ψ_0. □

Lemma 2.12. *Let $q : V \to \mathbb{Z}/2\mathbb{Z}$ be a non degenerate quadratic form where $dimV = 2m$. Then q is equivalent to $q_1+(m-1)q_0$ if, with respect to some basis, Arf(q) $=1$. Then form q is equivalent to mq_0 if Arf(q) $=0$.*

Proof. If $a_i, b_i,\ i=1,\cdots,m$, is a symplectic basis for V and if V_i is the subspace spanned by a_i, b_i, let ψ_i denote the restriction of q onto V_i. it is obvious that $q=\sum\psi_i$, where each ψ_i is equivalent to either q_0 or q_1. By the previous lemma, $2q_0=2q_1$, so q is equivalent to either mq_0 or $q_1+(m-1)q_0$, but Arf$(mq_0)=0$, and Arf$(q_1+(m-1)q_0)$, which implies the result. □

To complete the study of non-degenerate quadratic forms over $\mathbb{Z}/2\mathbb{Z}$, it remains to show that $\varphi_1 = q_1+(m-1)q_0$ and $\varphi_0 = mq_0$ are not equivalent. To show this we prove the following lemma:

Lemma 2.13. *The quadratic form φ_1 sends a majority of elements of V to $1\in\mathbb{Z}/2\mathbb{Z}$, while φ_0 sends a majority of elements to $0\in\mathbb{Z}/2\mathbb{Z}$.*

Proof. Proof by induction: (Case m=1), is trivial. Given a non-degenerate quadratic form φ on V, let $p(\varphi) = \#$ of v such that $\varphi(v)=1$, similarly let $n(\varphi)=\#$ of v such that $\varphi(v)=0$. Since, $\dim(V)=2n$ and can be given a symplectic basis we can conclude $n(\varphi)+p(\varphi)=2^{2m}$. The functions p and n satisfy the the identities $p(\varphi+q_0)=3p(\varphi)+q_0(v)$ and $n(\varphi+q_0)=3n(\varphi)+q_0(v)$ Set $r(\varphi)=p(\varphi)-n(\varphi)$. then $r(\varphi+q_0)=2r(\varphi)$, so that if $r(\varphi)>0$ then $r(\varphi+q_0)>0$ and if $r(\varphi)<0$ then $r(\varphi+q_0)<0$. It follows, since $r(q_1)=2$ and $r(q_0)=-2$, that $r(q_1+(m-1)q_0)>0$, and $r(mq_0)<0$, which proves the lemma. □

Corollary 2.14. *If q is a non-degenerate quadratic form, then Arf(q) $=1$ if and only if q sends a majority of elements of V to $1\in\mathbb{Z}/2\mathbb{Z}$. In particular, the Arf-invariant is well-defined.*

Since r in the above proof is an invariant, it follows thath $q_1+(m-1)q_0$ is not equivalent to mq_0. Thus we have reproved a Theorem of Arf:

Theorem 2.15 (C.Arf 1941)**.** *Two non-degenerate quadratic forms on a $\mathbb{Z}/2\mathbb{Z}-$vector space V of finite dimension are equivalent if and only if they have the same Arf-invariant.*

2.2. **The Arf-invariant of a knot.**

Knot theory provides a important example of where a 'natural' quadratic form arises. Let $k\subset S^3$ be a knot in the 3-sphere. Let F be a Seifert surface of genus-g and S its Seifert matrix in a fixed basis of the group $H_1(F;\mathbb{Z})$. The unimodular skew-symmetric form $I=S^T-S$ is the intersection form of the surface F. The for $Q=S+S^T$ is symmetric; it is even and has odd determinant because $Q=I\ mod\ 2$. Define a quadratic form $q : H_1(F;\mathbb{Z}/2\mathbb{Z})\to\mathbb{Z}/2\mathbb{Z}$ by the formula:

$$q(x)=\frac{1}{2}Q(x,x)\ mod\ 2.$$

Note that $q(x) = S(x,x)\ mod\ 2$. Its associated bilinear for is $I = Q\ mod\ 2$ since

$$\begin{aligned} q(x+y) - q(x) - q(y) &= S(x+y, x+y) - S(x,x) - S(y,y) \\ &= S(x,y) + S(y,x) \\ &= (S + S^T)(x,y) = Q(x,y) \end{aligned}$$

Lemma 2.16. *The Arf-invariant Arf(q) of the quadratic form* $\frac{1}{2}Q(x,x)\ mod\ 2$ *only depends on the knot* $k \subset S^3$ *and not on the choices in its definition.*

We denote the knot invariant Arf(q) as Arf(k) and call it the *Arf-invariant* of the knot k.

Proof. We only need check that Arf(q) is well-defined up to $S-$equivalence. Elementary enlargements and reductions replace Seifert matrices S by

$$\begin{bmatrix} & & & a_1 & 0 \\ & S & & \vdots & \vdots \\ & & & a_{2g} & 0 \\ b_1 & \cdots & b_{2g} & c & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Now by elementary row and column operations, we can make $c = 0$, and $a_i + b_i = 0$ for all $i = 1, \cdots, 2_g$. Then $Q' = S' + (S')^T$ is of the form

$$\begin{bmatrix} & & & 0 & 0 \\ & S+S^T & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

so a symplectic basis for $Q = S + S^T\ mod\ 2$ can be completed to a symplectic basis for $Q' = S' + (S')^T\ mod\ 2$ so that $\text{Arf}(q) + \text{Arf}(q_0) = \text{Arf}(q)\ mod\ 2$. ☐

Theorem 2.17. *Let k be a knot. The Arf-invariant of k is related to the Alexander polynomial by*

$$Arf(k) = \begin{cases} 0 & if \quad \Delta_k(-1) \equiv \pm 1\ mod\ 8 \\ 1 & if \quad \Delta_k(-1) \equiv \pm 3\ mod\ 8 \end{cases}$$

Also,

$$Arf(k) = \frac{1}{2}\Delta_k''(1)\ mod 2$$

Theorem 2.18 (Fox-Milnor)**.** *If k is a slice knot, then the (Conway-normalized) Alexander polynomial of k is of the form* $f(t)f(t^{-1})$*, where f is a polynomial with integer coefficients.*

Corollary 2.19. *If k is a slice knot then* $Arf(k) = 0$

3. Rokhlin's Theorem

3.1. **Characteristic Surfaces.**

Definition 3.1. Given any closed oriented 4−manifold M the intersection form Q_M is the pairing

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$
$$([A],[B]) \mapsto A \cdot B$$

Similarly, we can define $Q_M([A],[B])$ to be equal to $\langle PD_X^{-1}(A), B\rangle$ or $Q_M([A],[B]) = ([A] \smile [B])[M]$. Since the cup product is symmetric and bilinear so is Q_M.

We will take M to be simply-connected unless stated otherwise.

Definition 3.2. A closed oriented surface F smoothly embedded in M is called characteristic if

$$F \cdot x = x \cdot x \ mod\ 2 \ \text{ for all } x \in H_2(M;\mathbb{Z}).$$

Let $e_1, \cdots, e_n$ be a basis in $H_2(M)$ then Q_M is given by a matrix $a_{ij} = e_i \cdot e_j$. As a consequence a surface $F = \sum \varepsilon_i e_i$ is characteristic if and only if

$$\sum_{j=1}^{n} a_{ij}\varepsilon_j = a_{ii} \ mod\ 2 \ \text{ for all } \ i = 1, \cdots, n$$

With each characteristic surface $F \subset M$, one can associate a quadratic form

$$\widetilde{q} : H_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z},$$

and the $\mathrm{Arf}(M,F) = \mathrm{Arf}(\widetilde{q})$.

Now we give the punch line of this section:

Theorem 3.3 (Rokhlin's Theorem). *Let M be a simply-connected oriented closed smooth 4−manifold, and F a closed oriented surface smoothly embedded in M. If F is characteristic then*

$$\frac{1}{8}(\sigma(M) - F \cdot F) = Arf(M,F) \ mod\ 2.$$

Corollary 3.4 (Kervaire-Milnor). *If F in Theorem is a 2-sphere, $H_1(F;\mathbb{Z}/2\mathbb{Z})$ vanishes and $Arf(M,F) = 0$.*

The following corollary is obtained from taking F to be empty.

Corollary 3.5 (Rokhlin). *If M is a spin 4−manifold then $\sigma(M) \equiv 0 \ mod\ 16$*

3.2. **The definition of $\widetilde{q}$.**

Let F be a closed oriented characteristic surface smoothly embedded in M. Suppose that a homology classes $\gamma \in H_1(F;\mathbb{Z}/2\mathbb{Z})$ is realized by an embedded circle $\gamma \subset F$. Since $H_1(M;\mathbb{Z}) = 0$, γ bounds a connected orientable surface D embedded in M such that $int(D)$ is transversal to F. We may deform D slightly to anew surface D' so that $\gamma' = \partial D'$ is a curve in F obtained by shifting ∂D inside F so that $\partial D \cap \partial D' = \varnothing$. One may assume that D and D' intersect transversely. We define

$$\widetilde{q}(\gamma) = D \cdot D' + D \cdot F \ mod\ 2$$

where by $D \cdot D'$ and $D \cdot F$ we mean the intersection numbers of $int(D)$ with $int(D')$ and F, respectively.

Lemma 3.6.

$$\widetilde{q}(\gamma) = D \cdot D' + D \cdot F \ mod\ 2$$

gives a well-defined quadratic form

$$\widetilde{q} : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

whose associated bilinear form is the mod 2 intersection form of the surface F

Example 3.7. Suppose that a 3-sphere Σ embedded in M and separates the surface F into two pieces, $F = F' \cup \mathbb{D}^2$, where $F' \subset \Sigma$ is a Sefert surface of a knot $k \in \Sigma$. Then we have two quadratic forms,

(1) $q : H_1(F'; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, the quadratic form of a surface F

(2) $\widetilde{q} : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, defined above.

Claim 3.8. The inclusion induced isomorphism $\varphi : H_1(F'; \mathbb{Z}/2\mathbb{Z}) \to H_1(F; \mathbb{Z}/2\mathbb{Z})$ makes the following diagram commute:

$$\begin{array}{ccc} H_1(F'; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\quad q \quad} & \mathbb{Z}/2\mathbb{Z} \\ \Big\downarrow \varphi & & \Big\downarrow id \\ H_1(F; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\quad \widetilde{q} \quad} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

Proof of claim. Let $\gamma \subset F'$ be an embedded circle in F'. Choose an orientable embedded surface D with $\partial D = \gamma$ such that $D \cap D^2 = \varnothing$ (simply take D equal to a Seifert surface of γ inside Σ and push off D^2.) Then $D \cdot F = D \cdot F' = lk(\gamma, k)\ mod\ 2$. Let $N(\gamma)$ be a tubular neighborhood of γ in Σ. Since F' is a Seifert surface of the knot k, the intersection $\partial N(\gamma) \cap F'$ is homologous to k via the surface $F' \setminus \mathrm{int}(N(\gamma) \cap F')$. This implies that $[k] = [\partial N(\gamma) \cap F'] \in H_1(\Sigma \setminus \mathrm{int}(N(\gamma)); \mathbb{Z}) = \mathbb{Z}$. Therefore, $D \cdot F = lk(\gamma, k) = lk(\gamma, \partial N(\gamma) \cap F') = 0\ mod\ 2$. Thus $\widetilde{q}(\gamma) = D \cdot D' = lk(\gamma, \gamma') = lk(\gamma, \gamma^+) = q(\gamma)\ mod\ 2$ where γ^+ is a (positive) push-off of γ. □

Proof of Lemma. We first check that the number $\widetilde{q}(\gamma)$ mod 2 is independent of the choice of D. Let D_1 and D_2 be a two choice for D, and let $S = D_1 \cup_\gamma D_2$ and for simplicity we will assume S is smoothly embedded. Let $S' = D_1' \cup D_2'$, then $S \cdot S = S \cdot S' = D_1 \cdot D_1' + D_2 \cdot D_2'\ mod\ 2$. Since F is characteristic, $S \cdot S = S \cdot F\ mod\ 2$, so we get $D_1 \cdot D_1' + D_2 \cdot D_2' = D_1 \cdot F + D_2 \cdot F\ mod\ 2$ and $D_1 \cdot D_1' + D_1 \cdot F = D_2 \cdot D_2' + D_2 \cdot F\ mod\ 2$. Thus $\widetilde{q}(\gamma)$ is independent of the choice of D. Since any two homotopic closed simple curve on F are isotopic, $\widetilde{q}(\gamma)$ only depends on the homotopy class of γ, and hence defines a map $\widetilde{q} : \pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$.
Let $\gamma_1 * \gamma_2$ denote a product of loops γ_1 and γ_2, then we claim that

$$\widetilde{q}(\gamma_1 * \gamma_2) = \widetilde{q}(\gamma_1) + \widetilde{q}(\gamma_2) + \gamma_1 \cdot \gamma_2\ mod\ 2,$$

where $\gamma_1 \cdot \gamma_2$ is the intersection modulo 2 of the homology classes represented by γ_1 and γ_2. Since $\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_2\ mod\ 2$, the formula above implies that $\widetilde{q}(\gamma_1 * \gamma_2) = \widetilde{q}(\gamma_2 * \gamma_1)$ and that the map $\widetilde{q} : \pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$ factors through $H_1(F; \mathbb{Z})$ and $H_1(F; \mathbb{Z}/2\mathbb{Z})$.
Thus, to compelete the proof we just need to check the formula $\widetilde{q}(\gamma_1 * \gamma_2)$. For simplicity, let the curves γ_1 and γ_2 intersect transversely at one point, and let D_1 and D_2 be the surfaces that

the curves bound as in the definition of $\widetilde{q}$. Let γ be a smooth connectd sum loop representing $\gamma_1 * \gamma_2$. We get a bounding surface D for γ from $D_1 \cup D_2$ and the curved triangles T_1 and T_2. Push γ off in the direction of a normal field of γ extending the normal fields on γ_1 and γ_2. Then γ and its push-off will link, which indicates that $D \cdot D' = D_1 \cdot D_1' + D_2 \cdot D_2' + 1 \; mod \; 2$ □

Lemma 3.9. *Arf(M, F) only depends on homology class $[F] \in H_2(M; \mathbb{Z}/2\mathbb{Z})$.*

This upcoming proof of Rokhlin's Theorem was originally given by Andrew Casson a former University of Texas Professor.

Proof of Rokhlin's Theorem. Let us consider the manifold $M \# \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$. Its intersection form is odd and indefinite, hence isomorphic to the form $p \cdot (+1) \oplus q \cdot (-1)$ with $p = b_+(M) + 1$ and $q = b_-(M) + 1$. By Wall's theorem, there exists a $k \geq 0$ such that $(M \# \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2) \# k \cdot (S^2 \times S^2)$ is diffeomorphic to $(p \cdot \mathbb{C}\mathrm{P}^2 \# q \cdot \overline{\mathbb{C}\mathrm{P}}^2) \# k \cdot (S^2 \times S^2)$. Since

$$(S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2 = \overline{\mathbb{C}\mathrm{P}}^2 \# 2 \cdot \mathbb{C}\mathrm{P}^2, \; (S^2 \times S^2) \# \overline{\mathbb{C}\mathrm{P}}^2 = \mathbb{C}\mathrm{P}^2 \# 2 \cdot \overline{\mathbb{C}\mathrm{P}}^2,$$

We have that for some l_1 and l_2,

$$M \# l_1 \cdot \mathbb{C}\mathrm{P}^2 \# l_2 \cdot \overline{\mathbb{C}\mathrm{P}}^2 = a \cdot \mathbb{C}\mathrm{P}^2 \# b \cdot \overline{\mathbb{C}\mathrm{P}}^2,$$

where $a = l_1 + b_+(M)$ and $b = l_2 + b_-(M)$. Let $\eta \in H_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$ and $\widetilde{\eta} \in H_2(\overline{\mathbb{C}\mathrm{P}}^2) = \mathbb{Z}$ be the generators represented by the embedded 2-spheres $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$. Then $\eta \cdot \eta = 1$ and $\overline{\eta} \cdot \overline{\eta} = -1$. If a class F is characteristic in $H_2(M)$ then the class $F_c = F + l_1 \cdot \eta + l_2 \cdot \overline{\eta}$ is characteristic in $M \# l_1 \cdot \mathbb{C}\mathrm{P}^2 \# l_2 \cdot \overline{\mathbb{C}\mathrm{P}}^2$. The property of being characteristic is preserved under diffeomorphism, therefore, the image of F_c in $a \cdot \mathbb{C}\mathrm{P}^2 \# b \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is characteristic. Both the Arf-invariant and signature are additive with respect to conenected sums of manifolds and characteristic surfaces. Therefore, if the Rokhlin equality holds true for any of the following 3 pairs $(M_1, F_1), (M_2, F_2)$, and $(M_1 \# M_2, F_1 \cup F_2)$, it is true for the third one. We note that, $\sigma(\mathbb{C}\mathrm{P}^2) - \eta \cdot \eta = 0 = \mathrm{Arf}(\mathbb{C}\mathrm{P}^2, \eta)$ and $\sigma(\overline{\mathbb{C}\mathrm{P}}^2) - \overline{\eta} \cdot \overline{\eta} = 0 = \mathrm{Arf}(\overline{\mathbb{C}\mathrm{P}}^2, \overline{\eta})$. Moreover, both the Arf-invariant and signature both change signs if the orientation changes. Therefore, Rokhlin's theorem need only be checked for characteristic surfaces in $\mathbb{C}\mathrm{P}^2$.

If $\eta \in H_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$ is a generator represented by the embedded 2-sphere $\mathbb{C}\mathrm{P}^1$, then a class $s \cdot \eta \in H_2(\mathbb{C}\mathrm{P}^2)$ is characteristic if and only if and only if s is odd. The complex curve

$$C = \{[x_0 : x_1 : x_2] : x_0 x_1^{s-1} + x_2^s\} \subset \mathbb{C}\mathrm{P}^2$$

is homeomorphic to S^2 and represents the class $s \cdot \eta$ see below. It is smoothly embedded in $\mathbb{C}\mathrm{P}^2$ expect possibly at the point $[1 : 0 : 0]$. Let B be the $\mathbb{D}^4$ of the radius $\varepsilon > 0$ centered at $[1 : 0 : 0]$. In the affine plane $x_0 = 1$ the intersection $\partial B \cap C$ is given by the equations $x_0 x_1^{s-1} + x_2^s, \; |x_1|^2 + |x_2|^2 = \varepsilon^2$. Therefore, $\partial B \cap C \subset \partial B = S^3$ is the $(s, s-1)-$torus knot $k_{s,s-1}$. Let S be a Seifert surface in ∂B with the boundary curve $\partial B \cap C$, then the surface $F = (C \setminus (C \cap \mathrm{int} B)) \cup S$ represents the class $s \cdot \eta$. An easy calculation using the identification of the quadratic form q and $\widetilde{q}$ in the example above shows that

$$\begin{aligned}\mathrm{Arf}(\mathbb{C}\mathrm{P}^2, s\cdot\eta) &= \mathrm{Arf}(k_{s,s-1}),\\ &= (s^2-1)((s-1)^2-1)/24 \; mod\; 2\\ &= (1-s^2)/8 \; mod\; 2,\\ &= \frac{1}{8}(\sigma(\mathbb{C}\mathrm{P}^2) - s\eta\cdot s\eta)\; mod\; 2.\end{aligned}$$

□

We introduce a lemma that was used in the previous proof:

Lemma 3.10. *The complex curve C in $\mathbb{C}\mathrm{P}^2$ given by the equation $x_0x_1^{s-1} + x^s = 0$ is homeomorphic to S^2 and represents the homology class $s\cdot[\mathbb{C}\mathrm{P}^1] \in H_2(\mathbb{C}\mathrm{P}^2)$.*

3.3. **Representing homology classes by surfaces.** Let M be a simply-connected oriented closed smooth 4−manifold. It is known that every homology class of $H_2(M)$ can be represented by a smoothly embedded surface F. The following is one of the most intriguing problem in 4-dimensional topology: given a class $[u] \in H_2(M,)$ what is the minimal genus of $F \subset M$ representing u? The class u is said to be spherical if it can be represented by an embedded 2-sphere. Next we show a quick application of Rokhlin's Theorem to obstruct a homology class in $\mathbb{C}\mathrm{P}^2$ from being a 2-sphere.

Example 3.11. Note that $H_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$, so let $\eta \in H_2(\mathbb{C}\mathrm{P}^2)$ be the generator of the infinite cyclic group. We know that η can be represented by $[\mathbb{C}\mathrm{P}^1] \subset \mathbb{C}\mathrm{P}^2$ which is a 2-sphere. We will use Rokhlin's Theorem to show that the homology class $3\eta \in H_2(\mathbb{C}\mathrm{P}^2)$ is aspherical.Suppose that 3η is sphreical for sake of contradiction. By Kervaire-Milnor we know that if 3η is sphereical the it must have Arf-invariant equal to 0. We now calculate the $\mathrm{Arf}(\mathbb{C}\mathrm{P}^2, 3\eta)$,

$$\frac{1}{8}(\sigma(\mathbb{C}\mathrm{P}^2) - (3\eta\cdot 3\eta)) = \frac{1}{8}(1-9) = \frac{-8}{8} = -1 \equiv 1 \; mod\; 2 \neq \mathrm{Arf}(\mathbb{C}\mathrm{P}^2, 3\eta) = 0$$

Therefore, we see that the class 3η is aspherical.

3.4. **The Rokhlin invariant.** Let Σ be an oriented integral homology 3-sphere. Then there exists a smoothly simply-connected 4-manifold W with even intersection form such that $\partial W = \Sigma$. Then the signature of W is divisible by 8, and

$$\mu(\Sigma) = \frac{1}{8}\sigma(W)\; mod\; 2$$

is independent of choice of W. We call $\mu(\Sigma)$ the Rokhlin invariant of Σ. Suppose that M is a smooth simply-connected oriented 4−manifold with $\partial M = \Sigma$; we do not even assume a intersection form. Suppose that M has a spherical characteristic surface then,

$$\mu(\Sigma) - \frac{1}{8}(\sigma(M) - F\cdot F)\; mod\; 2.$$

Now to check the formula, form a smooth closed manifold $X = M \cup_\Sigma (-W)$. Then F is spherical characteristic surface in X, and

$$\begin{aligned}\frac{1}{8}(\sigma(M) - F\cdot F) - \mu(\Sigma) &= \frac{1}{8}(\sigma(M) - F\cdot F) - \frac{1}{8}\sigma(W)\\ &= \frac{1}{8}(\sigma(X) - F\cdot F) = 0\; mod\; 2.\end{aligned}$$

Example 3.12. As as concrete exaple, 1-surgery on the trefoil $K = 3_1$, denoted $S^3_1(K)$, has Rokhlin invariant $\mu(S^3_1(K)) = 1$ because $S^3_1(3_1)$ is known to bound the simply-connected smooth spin 4-manifold obtained from $\mathbb{D}^4$ by attaching eight 2−handles along a framed link with linking matrix $-E_8$.

4. Current Research involving Arf-Invariant, and Rokhlin's Theorem

Knot invariants from branched covers of 4-manifolds" Alexandra Kjuchukova discussed two knot $k \subset S^3$ invariants defined in terms of branched covers of 4-manifolds. The first is the Ξ_p invariant, which comes from irregular p-fold covers of B^4. The second one, j, is a new concordance invariant of framed knots in 3-manifolds. As applications, I will illustrate how to use Ξ_p to obstruct ribbonness for twist knots; and I will relate Ξ_p and j to other classical invariants. Some of this work is joint with Julius Shaneson.