Characteristic Classes, Principal Bundles, and Curvature

Jacob V. Gaiter

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Contents

1 Preface

These are the lecture notes for my 2024 Summer Minicourse on Characteristic classes, taking place at UT Austin. Section 2.1 is completely optional, and serves purely as motivation. Sections 2.2- 4.4 serve as the algebraic-topological portion of the course, and Sections 5 serves as the differential geometric portion. These two portions are largely independent from one another and can be read separately.

The first section will provide some light motivation for the use of vector bundles in differential topology. For a more comprehensive coverage of the differential geometric aspects covered in this section (and in the rest of the notes), see [7] or [6]. For the differential topological aspects, see [4]. Later sections will rely on the cup product structure on cohomology, Poincaré duality, and the cross product. For these topics, see [2], [3], or [5]. We will also make use of results regarding the cohomology of fiber bundles, in particular the Leray-Hirsch theorem, I recommend taking these as a black box, but the motivated reader should consult [3].

Throughout these notes let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. We may freely switch between notions in smooth and continuous categories, and the correct category should (hopefully) be clear from context.

2 Introduction to Vector Bundles

2.1 Why Vector Bundles?

The key concept of differential calculus is the idea of linearization: given a nice enough function $f : (a, b) \to \mathbb{R}$, there is a unique linear function $df(p)$ (given by a number, typically written $f'(p)$) which best approximates f near $p \in (a, b)$, meaning $f(p + tv) \approx f(p) + tdf(p)v$ for small v. This means we have linearized f near p.

Of course, on an open subset U of $\mathbb{R}^n$ we can employ identitical ideas to find linearizations of maps $f : U \to \mathbb{R}^m$ and classical theorems like the inverse function theorem and the implicit function theorem tell us how we can understand properties of f by understanding properties of df.

The philosophy of differential topology tells us that we should try to understand global behavior of a map $f : M \to N$ by patching together local information. While the idea of the differential as a map $U \times \mathbb{R}^n \to \mathbb{R}^m$ becomes less meaningful on a general smooth manifold, we can reinterpret it as a map $TU \to T\mathbb{R}^m$ or as a bundle map $TU \to f^*T\mathbb{R}^m$. This globalizes to the vector bundle morphism $Tf : TM \to f^*TN$. One can think of the space of nearby maps to f as sections of the vector bundle f^*TN, and hence one can turn non-linear partial differential equations on $\mathrm{Hom}(M, N)$ into partial differential equations on vector bundles related to f^*TN, which are simpler to deal with.

The pointwise local properties of this map tell us something about the global behavior of the function, for instance, the constant rank theorem says that if Tf_p is of constant rank for all $p \in M$ then $f^{-1}(p)$ is an embedded submanifold of M.

Another useful concept of linearization coming from vector bundles is that of a normal bundle. Given an embedded submanifold $\iota : M \to N$, the differential yields the following short exact sequence of vector bundles:

$$0 \to TM \to \iota^*TN \to N_{M/N} \to 0$$

where $N_{M/N,p} := T_pN/T_pM$. Infinitesimally, the normal bundle tells us about how to move away from M inside of N. This is manifest in the content of the tubular neighborhood theorem:

Theorem 2.1. *Let $\iota : M \to N$ be a embedded submanifold, then there is an open neighborhood of M in N which is diffeomorphic to a convex neighborhood of the zero section of $N_{M/N}$, via a diffeomorphism preserving* 0.

This allows us to capture topological information about how M sits inside N by understanding $N_{M/N}$.

Remark. Since M embeds as 0 inside of E for any vector bundle $E \to M$, every vector bundle is the normal bundle of M inside some other smooth manifold.

This idea of linearization allows us to compute self intersections, even if we only understand the normal bundle.

Proposition 2.2. *If* $M \to N$ *is an embedded closed submanifold of middle dimension, the self intersection of* M *in* N *is equal to that of* M *in* $N_{M/N}$.

Exercise 1. Prove proposition 1.2.

It turns out that this self intersection is a homological quantity, and can be computed using characteristic classes.

We can see that if $N_{M/N} \cong M \times \mathbb{R}^m$ then M has zero self intersection in N. The self intersection somehow quantifies how twisted $N_{M/N}$ is. The idea behind characteristic classes is to quantify how twisted (i.e. how far from trivial) various vector bundles are using algebraic topology.

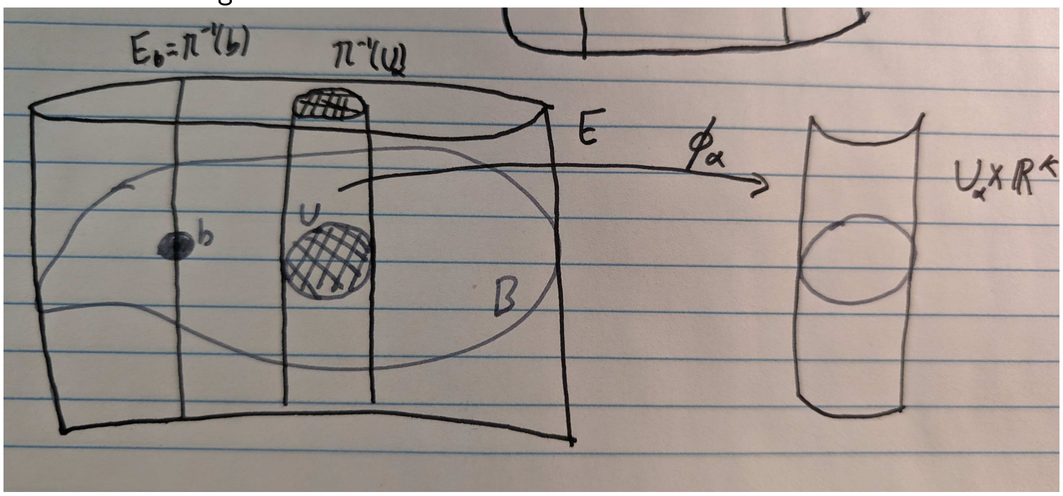
2.2 Formal Definitions and Operations on Vector bundles

The idea of vector bundles, and bundles more generally, is to capture the notion of smoothly (or continuously) varying families of objects which are locally trivial, i.e. when we zoom in close enough in the base, the family looks like the trivial family of a reference fiber. This family, denoted typically by $\pi : E \to B$ consists of E, B smooth manifolds (or topological spaces) and $\pi : E \to B$ a smooth (continous) map. Such a triple is called a rank k vector bundle over B if there is an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and a family of diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{K}^k \times U_\alpha$ making the diagram:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & \mathbb{K}^k \times U_\alpha \\ {\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle pr_2} & \\ U_\alpha & & \end{array}$$

(all this means is that φ_α interchanges π with pr_2) and that the transition maps $\varphi_\alpha^{-1} \circ \varphi_\beta : \mathbb{K}^k \times (U_\alpha \cap U_\beta) \to \mathbb{K}^k \times (U_\alpha \cap U_\beta)$ is given by a function $U_\alpha \cap U_\beta \to GL_k(\mathbb{K})$, where $GL_k(\mathbb{K})$ is the space of invertible $k \times k$ matrices with entries in $\mathbb{K}$. E is referred to as the total space, B as the base space, and π the projection map.

Figure 1: Vector bundle and a local trivialization



Exercise 2. Prove that for each $p \in B$, $\pi^{-1}(p)$ carries a natural vector space structure, and is isomorphic to $\mathbb{K}^k$.

Remark. The product $pr_2 : \mathbb{K}^n \times X \to X$ is a rank n vector bundle over X with a global trivialization given by the identity, this is called the trivial rank n vector bundle over X.

Example 1. The tangent bundle $TM \to M$ to a smooth manifold M^n is a prototypical example of a vector bundle.

Exercise 3. Prove, using your favorite definition of the tangent bundle, that $TM \to M$ is a vector bundle.

Example 2. Consider projective space: $\mathbb{P}^n(\mathbb{K}) := (\mathbb{K}^{n+1} \backslash \{0\})/\mathbb{K}^*$. Geometrically, this is the space of one dimensional subspaces of $\mathbb{K}^{n+1}$. There is a tautological family of vector spaces parameterized by $\mathbb{P}^n(\mathbb{K})$, $\pi : \tau \to \mathbb{P}^n(\mathbb{K})$ given by associating to each element $[x] \in \mathbb{P}^n(\mathbb{K})$ the subspace that it spans, i.e. $\mathbb{K}x$. τ lives as a subspace of the trivial bundle $\mathbb{K}^{n+1} \times \mathbb{P}^n(\mathbb{K})$, as the set of elements $(v, [x])$ such that $v = \lambda x$ for some $\lambda \in \mathbb{K}$. This comes with natural local trivializations over the image of $\{(x_0, x_1, \ldots, x_n) | x_i \neq 0\}$ under the quotient map, denote this open set by U_i. Given a point $x = [(x_0, x_1, \ldots, x_n)] \in U_i$ we can write it as $x = [(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i})]$ so that a vector $v \in \tau_x$ can be written uniquely as $v = \alpha(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i})$. This yields a map from $\pi^{-1}(U_i) \to \mathbb{K} \times U_i$, $v = \alpha(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i}) \mapsto (\alpha, [x_0, x_1, \ldots, x_i, \ldots, x_n])$.

Example 3. Consider the trivial line bundle over $\mathbb{R}$, $\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This admits a $\mathbb{Z}$ action by $a(x, y) = ((-1)^a x, y + a)$. Denote $\rho : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $\rho' : \mathbb{R} \times \mathbb{R} \to (\mathbb{R} \times \mathbb{R})/\mathbb{Z}$ the projection maps. Since the $\mathbb{Z}$ action on $\mathbb{R} \times \mathbb{R}$ commutes with the projection $\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, there is unique map $(\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \xrightarrow{\mathrm{pr}_2} & \mathbb{R} \\ {\scriptstyle \rho'}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ (\mathbb{R} \times \mathbb{R})/\mathbb{Z} & \xrightarrow[\pi]{} & \mathbb{R}/\mathbb{Z} \end{array}$$

The quotient $\pi : (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ admits a vector bundle structure. Thinking more geometrically, one can picture a fundamental domain of $\mathbb{R} \times \mathbb{R}$ as a strip $\mathbb{R} \times [0, 1]$ and the identification is gotten by identifying $\mathbb{R} \times 0$ with $\mathbb{R} \times 1$ via $(x, 0) \mapsto (-x, 1)$. This yields the classic picture of the Möbius strip!

Exercise 4. Prove that $(\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a vector bundle.

Definition 2.3. Given a vector bundle (or any bundle for that matter) a helpful idea is the notion of a section (or historically, cross-section). Intuitively, a section of $\pi : E \to B$ is a continuous (smooth) choice of an element of the fiber E_b above each point $b \in B$. Formally, a section is a continous (smooth) map $\sigma : B \to E$ with $\pi \circ \sigma = \mathrm{Id}_B$.

Remark. Every vector bundle comes with a natural section given by the assignment of 0 to each point in B. This means that B always comes embedded in E as the image of the zero section. In the images of vector bundles, both here and elsewhere, you will find the zero section included.

The sections of a trivial bundle $\mathbb{K}^n \times B$ are given by maps $B \to \mathbb{K}^n$, i.e. by vector valued functions on B. In this way, we can think of sections of a vector bundle as "twisted" K^n valued functions on B.

Remark. Given two families of objects over a base B, $\pi, \pi' : E, E' \to B$, a morphism of these families is naturally defined as a morphism of the total spaces $E \to E'$ which covers the projections, i.e. the following diagram commutes

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

If the fibers have additional structure, we would ask that this morphism preserve that structure. This leads to the following definition.

Definition 2.4. Given $\pi, \pi' : E, E' \to B$ vector bundles over $\mathbb{K}$, a morphism of vector bundles $(\pi, E, B) \to (\pi', E', B)$ (often denoted by $E \to E'$) is a map $T : E \to E'$ with $\pi' \circ T = \pi$ such that the restriction to each fiber $E_p \to E'_p$ is a linear map.

By applying operations that are continuous to each of the fibers of a bundle we can create new vector bundles from old.

Example 4. Let $\pi : E \to B$ and $\rho : E' \to B$ be vector bundles over B of rank k and r respectively. We can form their fiberwise direct sum (the so called Whitney sum) $E \oplus E' := \coprod_{b \in B} \pi^{-1}(b) \oplus \rho^{-1}(b)$. As a set this comes with a natural map $\pi_{E \oplus E'} : E \oplus E' \to B$ by taking $\pi^{-1}(b) \oplus \rho^{-1}(b) \ni (v, w) \mapsto b$.

Theorem 2.5. *The direct sum* $\pi_{E \oplus E'} : E \oplus E' \to B$ *carries a natural vector bundle structure.*

Proof. We can naturally topologize a vector bundle by enforcing the local trivializations to be homeomorphisms on the image. That is, a vector bundle E is given the unique topology in which $\varphi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{K}^k \times U_\alpha$ is a homeomorphism for each U_α the domain of a local trivialization.. This means we can toplogize $E \oplus E'$ by specifying a collection of local trivializations. Given local trivializations $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ and $\{(V_\gamma, \psi_\gamma)\}_{\gamma \in \mathcal{A}'}$ for E and E' respectively, we can specify a local trivialization $\Phi_{\alpha\gamma} : \pi^{-1}_{E \oplus E'}(U_\alpha \cap V_\gamma) \to \mathbb{K}^{k+r} \times U_\alpha \cap V_\gamma$ by $\pi^{-1}(b) \oplus \rho^{-1}(b) \ni (v, w) \mapsto (\mathrm{pr}_1(\varphi_\alpha(v)), \mathrm{pr}_1(\psi_\gamma(v)), b)$. The inverse map $\Phi^{-1}_{\alpha\gamma}$ is simply given by $(v, w, b) \mapsto (\varphi^{-1}_\alpha(v, b), \psi^{-1}_\gamma(w, b))$ so that the transition maps $\Phi^{-1}_{\alpha\gamma} \circ \Phi_{\beta\delta}$ are given by $(v, w, b) \mapsto (\varphi^{-1}_\alpha \circ \varphi_\beta(v, b), \psi^{-1}_\gamma \circ \psi_\delta(w, b))$ which is valued in $GL_{k+r}(\mathbb{K})$ as a block matrix. ☐

Remark. A simpler, though more opaque and complicated in the smooth case, way to prove the preceding theorem is to realize that $E \oplus E'$ is actually the fibered product appearing in the following diagram:

$$\begin{array}{ccc} E \oplus E' & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ E' & \xrightarrow[\rho]{} & B \end{array}$$

And the local trivializations yield isomorphisms between restrictions of the previous diagram and:

$$\begin{array}{ccc} \mathbb{K}^{r+k} \times (U_\alpha \cap V_\gamma) & \longrightarrow & \mathbb{K}^k \times (U_\alpha \cap V_\gamma) \\ \downarrow & & \downarrow \\ \mathbb{K}^r \times (U_\alpha \cap V_\gamma) & \longrightarrow & U_\alpha \cap V_\gamma \end{array}$$

Exercise 5. Show that the maps $E \oplus E' \to E$ and $E \oplus E' \to E'$ are bundle morphisms, and that there are two short exact sequences of vector bundles (i.e. diagrams of vector bundles which are exact on fibers):

$$0 \to E \to E \oplus E' \to E' \to 0$$
$$0 \to E' \to E \oplus E' \to E \to 0$$

Remark. In general, a short exact sequence of vector bundles would be written as

$$0 \to E \to F \to E' \to 0.$$

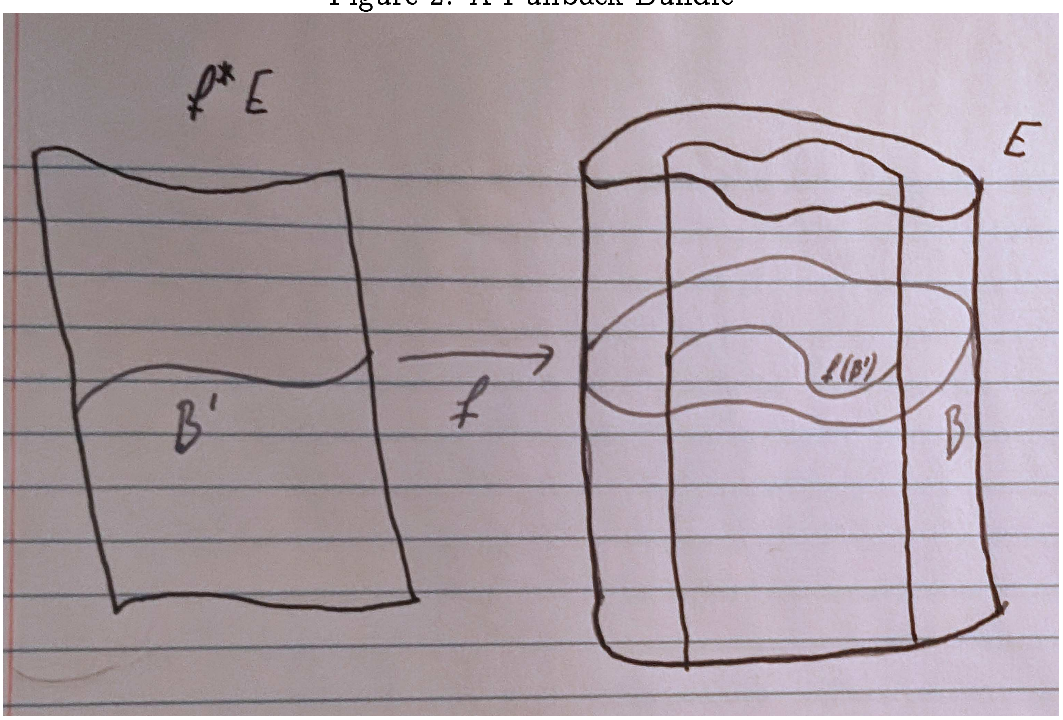
The preceding example is called a split short exact sequence as here we can write $F \cong E \oplus E'$ with a map which respects the inclusion $E \to F$ and projection $F \to E'$. In the appropriate topological and smooth categories, i.e. where we consider vector bundles with continuous or smooth transition maps and morphisms (defined over the correct domain of course), every short exact sequence is split due to the existence of Euclidean and Hermitian metrics, over $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ respectively. This is not the case in places like complex geometry, where splittings in the smooth category almost never respect the holomorphic structure.

Another way of yielding new vector bundles from old is that of the pullback bundle. Given $\pi : E \to B$ a vector bundle over B and $f : B' \to B$, as a topological space we can form the pullback square:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \underset{f}{\longrightarrow} & B \end{array}$$

The space $f^*E = \{(v, b) \in E \times B' : f(v) = \pi(b)\}$ so that the projection $f^*E \to B'$ is given by the restriction of $E \times B' \to B'$ and the fiber $f^*E_{b'} = E_{f(b)}$; this is the most natural operation to do, we just associate to each point in the domain of f the fiber of the point that f takes it to.

Figure 2: A Pullback Bundle



Exercise 6. Prove that f^*E is a vector bundle over B'.

To a pair of vector bundles E and E', we can define $\mathrm{Hom}(E, E')$ as the total space of a vector bundle with fiber $\mathrm{Hom}(E, E')_p := \mathrm{Hom}(E_p, E'_p)$.

Exercise 7. Prove that $\mathrm{Hom}(E, E')$ is a vector bundle, and the space of vector bundle homomorphisms $E \to E'$ is naturally isomorphic (as a vector space) to the space of sections of $\mathrm{Hom}(E, E')$.

Remark. The structures given above indicate that one should form the category of vector bundles $\mathrm{Vect}(X)$ where the morphisms are given by morphisms of vector bundles, and that this category should carry some abelian like structure. However, the natural definition of the kernel of a morphism $F : E \to E'$ as the sub-space of E for which F vanishes need not be a vector bundle as the ranks of the fibers may jump when the rank of F changes. One could consider the category equipped with morphisms with constant rank, but this severely limits the applications of such a theory.

3 Oriented Vector Bundles and the Euler Class

3.1 Euclidean and Oriented Vector bundles

Thinking of vector bundles as families of vector spaces over our base, it makes sense to equip them with structures we often equip vector spaces with, which vary in a smooth/continous way. The first of these notions is that of an inner product space, which yields Euclidean vector bundles.

Definition 3.1. Let $\pi : E \to B$ be a vector bundle over $\mathbb{R}$. A Euclidean structure on E is the data of a smooth/continous map $E \times_\pi E \to \mathbb{R}$ (i.e. a collection of maps $E_p \times E_p \to \mathbb{R}$) which is an inner product on each fiber.

Remark. When B is paracompact, one can use a partition of unity argument to show that every real vector bundle admits a Euclidean structure, and that this choice is unique up to homotopy (in a precise sense).

Just as a we can orient a vector space, by choosing a class of positively oriented ordered bases, we can think about choosing a continous or smooth family of orientations.

Definition 3.2. Let $E \to B$ be a rank k real vector bundle. From E we can create the orientation bundle $\mathrm{Or}(E) \to B$ with

$$\mathrm{Or}(E)_p = \{(\nu_1, \dots, \nu_k) \in E_p^k | \{\nu_1, \dots, \nu_k\} \text{ is a basis for } E_p\} / \sim,$$

where $(v_1, \dots, v_k) \sim (u_1, \dots, u_k)$ if the matrix A with $u_i = \sum_j A_i^j v_j$ has $\det(A) > 0$. $\mathrm{Or}(E)_p$ is precisely the set of orientations of E_p.

Remark. Since every real vector space only has 2 orientations, the map $\mathrm{Or}(E) \to B$ is a double cover, often called the orientation double cover of E.

Remark. At this point, we could make a detour into the world of Stiefel-Whitney classes. Assuming B is connected, a nontrivial (i.e. connected) double cover defines a surjective homomorphism $w_1(E) : \pi_1(B) \to \mathbb{Z}/2\mathbb{Z}$ by the action of the fundamental group as the deck transformations of $\mathrm{Or}(B)$, the trivial double cover induces the trivial action on $\mathbb{Z}/2\mathbb{Z}$ since $\pi_1(B)$ acts trivially on the space of connected components of any cover. The Hurewicz theorem and universal coefficients then tell us that $\mathrm{Hom}(\pi_1(B), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(B, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong H^1(B, \mathbb{Z}/2\mathbb{Z})$, so that $w_1(E) \in H^1(B, \mathbb{Z}/2\mathbb{Z})$, but we won't say much more about Stiefel-Whitney classes.

Definition 3.3. An orientation for $E \to B$ is a section $B \to \mathrm{Or}(E)$, i.e. a continously varying choice of orientation on each E_p. E is said to be orientable if there exists an orientation.

Remark. Our previous remark tells us that if B is simply connected, E is necessarily orientable! One can make this connection through the language of Stiefel-Whitney classes, characteristic classes for not-necessarily-oriented real vector bundles.

Remark. In the case of real line bundles, the concept of orientation is much simpler. Let $L \to B$ be a real line bundle, i.e. $\mathrm{rk}(E) = 1$. The space of bases of L_p is $L_p \backslash 0$. This means that the orientation bundle is simply $(L \backslash (0(B)))/\mathbb{R}^+$. We can use this to see that the open Möbius strip is not orientable as a vector bundle over S^1. We can identify the quotient with the boundary of the closed Möbius strip, which is not a trivial double cover over S^1 and hence admits no section. In fact, a real line bundle is trivial if and only if it is orientable.

Exercise 8. Let $L \to B$ be a real line bundle. Prove that L is trivial if and only if it is orientable.

3.2 The Euler Class

In the case of smooth manifolds, one can make use of Poincaré duality to construct characteristic classes in a manifestly geometric way.

Definition 3.4. Let $E \to M$ be a rank k smooth oriented vector bundle over M an oriented closed manifold. The Euler class $e(M) \in H^k(M, \mathbb{Z})$ is defined by $PD(\sigma^{-1}(0))$ for $\sigma \in \Gamma(E, M)$ with σ intersecting the zero section of E transversely.

Proposition 3.5. *$e(E)$ is well defined, and satisfies $e(f^*E) = f^*e(E)$ for $f : M \to N$ a smooth map, and $e(E \oplus F) = e(E)e(F)$.*

Proof. Endow E with a Euclidean structure. We can then construct a sphere bundle over M with fibers the 1-pt compactification of E_p, denote this by $B(E) \to M$. (Think of this as taking the double of the unit disc bundle $D(E) \to M$). This space is a smooth oriented closed manifold with orientations coming from the fiber orientations and the base. Denote $\iota : M \to B(E)$ the zero section, thought of as a section of $B(E) \to M$. We then write $e(E) = \iota^*(PD(\sigma_*[M]))$. Any two sections $M \to E$ are isotopic since $\Gamma(E, M)$ is a vector space and hence contractible. As such the class $e(E)$ does not depend on our choice of section.

Assume that $f : M \to N$ is a smooth map and $\pi : E \to N$ is an oriented euclidean vector bundle. The space $M \times E \to M \times N$ by $\mathrm{id} \times \pi$ is a vector bundle over $M \times N$. A transversely vanishing section σ of E extends to a transversely vanishing section of $M \times E$ by $\tilde{\sigma} = \mathrm{id}_M \times \sigma$. Here, $\tilde{\sigma}^{-1}(0) = M \times \sigma^{-1}(0)$. Let σ' be a section of f^*E. Such a section can be extended to a transversely vanishing section $\tilde{\sigma}'$ of $M \times E$, by considering M as the image of $\Gamma : M \to M \times N, m \mapsto (m, f(m))$, and $f^*E \cong \Gamma^*(M \times E)$. As such $PD(\tilde{\sigma}'^{-1}(0)) = PD(\tilde{\sigma}^{-1}(0)) = 1 \times e(E)$ and

$$e(f^*E) = PD(\sigma'^{-1}(0)) = \Gamma^*(1 \times e(E)) = f^*e(E)$$

To prove the product formula, consider $\pi, \pi' : E, E' \to M$ two vector bundles over M. We can write $E \oplus E' = \Delta^*(E \times E')$ where $\Delta : M \to M \times M$ is the diagonal map, and $E \times E' \to M \times M$ is given the natural vector bundle structure. Given σ, σ' transversely vanishing sections of M and M' respectively, $\Sigma = \sigma \times \sigma'$ is a transversely vanishing section of $E \times E'$, with $\Sigma^{-1}(0) = \sigma^{-1}(0) \times \sigma'^{-1}(0)$. We may now conclude $e(E \times E') = e(E) \times e(E')$ so that

$$\begin{aligned} e(E \oplus E') &= e(\Delta^*(E \times E)) \\ &= \Delta^*(e(E \times E')) \\ &= \Delta^*(e(E) \times e(E')) \\ &= e(E)e(E') \end{aligned}$$

□

Remark. There are clearly issues with this definition, i.e. that it requires the technology of Poincaré duality, but it has very high intuitive content. We can see imediately that the Euler class gives an obstruction to finding a nonvanishing section of E.

Proposition 3.6. *Let $E \to M$ be an oriented vector bundle over the oriented, closed manifold M. If E admits a non-vanishing section, then $e(E) = 0$.*

Proof. Let σ be a non-vanishing section of M. Given σ a nonvanishing section we conclude that $\sigma'^{-1}(0) = \varnothing$ and hence $e(E) = 0$. □

Remark. Proposition 3.6 can actually be upgraded to an if and only if statement using obstruction theory, see [8]. Hence the Euler class gives the only obstruction to the existence of a non-vanishing section.

Since the cup product is dual to intersections, the Euler class allows us to compute self intersection numbers cohomologically.

Proposition 3.7. *If $S \to M$ is a closed, oriented, middle dimensional submanifold, the self intersection number of S, is given by the Poincaré dual of the Euler class, in equations:*

$$\#(S \cap S) = \langle e(N_S), [S] \rangle.$$

Proof. By the tubular neighborhood theorem, there is an open neighborhood of S which is diffeomorphic to N_S and hence a transverse pushoff of S can be produced by taking a transversely vanishing section $\sigma : S \to N_S$ and $\#(S \cap S) = \sigma(S) \cap S = PD(e(N_S)) \cap [S] = \langle e(N_S), [S] \rangle$. □

This allows us to state and prove the theorem that gives the Euler class its name:

Theorem 3.8. *If* M *is a closed oriented smooth manifold, then* $\chi(M) = \langle e(TM), [M] \rangle$. *Where* TM *is considered as an oriented vector bundle with orientation inherited from* M.

Proof. Let σ be a transversely vanishing section of TM. Poincaré-Hopf tells us that $\chi(M) = \#(\sigma^{-1}(0))$ and hence $\chi(M) = \langle e(TM), [M] \rangle$. □

From this we should interpret the Euler class as a generalized Euler characteristic, which tracks how "twisted" a space M is as the zero section of E. In future potions of the course, Chern-Weil theory will allow us to exhibit a representative of $e(N)$ as a differential form on M, yielding a connection between the topology of self intersection with the geometry of differential forms and curvature. This is an incarnation of the principle which yields the Gauss-Bonnet theorem, and its generalization the Chern-Gauss-Bonnet theorem.

3.3 Axiomatics of the Euler Class

The naturality properties of the Euler class from Proposition 3.5 allow us to think of the Euler class more categorically. To each smooth manifold M, we can associate the set of isomorphism classes of (oriented) vector bundles over M, with field $\mathbb{K}$, $\mathrm{Vect}_{\mathbb{K}}(M)$. This is set valued as any vector bundle over M is determined by the data of its trivializations over a basis for its topology. This assignment yields a contravariant functor $\mathrm{Vect}_{\mathbb{K}} : \mathsf{SmMan} \to \mathsf{Set}$ with $f \mapsto (E \mapsto f^*E)$, (It turns out that this functor is representable by a classifying space, at least in the topological category). The cohomology functor also yields such a functor $H^*(-, \mathbb{Z}) : \mathsf{SmMan} \to \mathsf{Set}$. The fact that $e(f^*E) = f^*e(E)$ means that the Euler class is a natural transformation between these two functors. A natural question is whether or not we can characterize the Euler class uniquely through these properties.

Exercise 9. Prove that there is a one to one correspondence between characteristic classes, i.e. an assignment of $k(E) \in H^*(B)$ for every isomorphism class of $\mathbb{K}$-vector bundles $E \to B$ which is natural $k(f^*E) = f^*k(E) \in H^*(B')$ for every map $f : B' \to B$, and natural transformations between the $\text{Vect}_{\mathbb{K}}$ functor and the cohomology functor $H^*(-)$.

Proposition 3.9. *The Euler class is the unique natural transformation between* $\text{Vect}_{\mathbb{K}}$ *and* $H^*(-, \mathbb{Z})$ *satisfying the following axioms:*

a) $e(E \oplus E') = e(E) \cup e(E')$

b) If $\bar{E}$ *is the orientation reversed copy of* E *then* $e(\bar{E}) = -e(E)$

c) The Euler class of $\tau \to \mathbb{P}^1(\mathbb{C})$ *has* $\langle e(\tau), [\mathbb{P}^1(\mathbb{C})]\rangle = -1$.

d) The Euler class' mod 2 reduction is the top Stiefel-Whitney class.

4 Complex Vector Bundles and Chern Classes

4.1 Hermitian Vector Bundles

Definition 4.1. Let V be a finite dimensional complex vector space. A Hermitian metric on V is a map $h : V \times V \to \mathbb{C}$ such that $h(\lambda v + u, w) = \lambda h(u, w) + h(v, w), h(v, u) = \overline{h(u, v)}$ for all $u, v, w \in V$ and $\lambda \in \mathbb{C}$ with the property that $h(u, u) \geq 0$ for all $u \in V$ and $h(u, u) = 0$ if and only if $u = 0$.

Hermitian metrics are the standard way to measure angles and lengths on a complex vector space.

Definition 4.2. Let $E \to B$ be a complex vector bundle. A Hermitian structure on E is a continuous choice of Hermitian metrics on each fiber, i.e. a map $E \times_\pi E \to \mathbb{C}$ such that the restriction to each fiber $E_p \times E_p \to \mathbb{C}$ is a Hermitian metric.

Exercise 10. Prove that for any B a paracompact space, and $E \to B$ a complex vector bundle, E admits a Hermitian structure.

Example 5. Any choice of Hermitian metric h on V a $\mathbb{C}$-vector space naturally induces a Hermitian metric on $\tau \to \mathbb{P}(V)$ as τ is a subbundle of $V \times \mathbb{P}(V)$, and $h : V \times V \to \mathbb{C}$ gives a metric on $V \times \mathbb{P}(V)$.

4.2 The First Chern class

Before setting out on defining the first Chern class, we must note a fundamental fact about complex line bundles.

Proposition 4.3. *Let* $L \to B$ *be a complex line bundle over* B*. Considered as a real vector bundle,* L *admits a natural choice of orientation.*

Proof. Let $(U_\alpha, \varphi_\alpha)$ be a system of (complex) local trivializations of L. Over each U_α choose the orientation $[(\varphi^{-1}(1), \varphi^{-1}(i))]$. Since the transition maps of $(U_\alpha, \varphi_\alpha)$ are valued in $GL_1(\mathbb{C}) \subset GL_2(\mathbb{R})^+$, all transition maps are of positive determinant and this choice of orientation globalizes. $\square$

Now, from any complex vector bundle $E \to B$ we naturally obtain a complex line bundle $\det(E) \to B$. There are several ways to define $\det(E)$. One way to think about it is $\det(E)_p = \wedge^{\dim E} E$. For L a complex line bundle $\det(L) = \bigwedge^1 L \cong L$.

Definition 4.4. We define the first Chern class of E to be $c_1(E) := e(\det(E))$.

Remark. When equipped with the opposite complex structure, gotten by replacing i with $-i$, this has the effect of changing the orientation by $(-1)^{\mathrm{rk}(E)}$ as there are $\mathrm{rk}(E)$ many minus signs introduced into our natural orientation. This means that for a line bundle, the natural orientation $[(e_1, -ie_1)]$ differs from the opposite $[(e_1, ie_1)]$ by -1. Hence $c_1(\bar{E}) = -c_1(E)$.

Proposition 4.5. *The first Chern class is natural, i.e. for* $f : B' \to B$, $c_1(f^*E) = f^*c_1(E)$*. Furthermore,* c_1 *satisfies a product operation,* $c_1(E \oplus E') = c_1(E) + c_1(E')$

Proof. As oriented real vector bundles, we have $\det(f^*E) \cong f^* \det(E)$ so that by naturality of the Euler class

$$c_1(f^*E) = e(\det(f^*E)) = e(f^* \det(E)) = f^*e(\det(E)) = f^*c_1(E)$$

For the product operation, note that $\det(E \oplus E') \cong \det(E) \otimes \det(E')$. In the "geometric" Euler class picture, we can see that given transversely vanishing sections σ, σ' of $\det(E)$ and $\det(E')$ respectively, $(\sigma \otimes \sigma')^{-1}(0) = \sigma^{-1}(0) \cup (\sigma')^{-1}(0)$ so that $e(\det(E) \otimes \det(E')) = e(\det(E)) + e(\det(E'))$ as desired. □

The first Chern class, by our previous observations measures the obstruction to finding a section of $\det(E)$. Such a section gives a kind of "complex orientation" of E.

Example 6. Consider the dual of the tautological line bundle over $\mathbb{P}^1(\mathbb{C})$, $\tau^* = \mathrm{Hom}(\tau, \mathbb{C} \times \mathbb{P}^1(\mathbb{C}))$. τ has a natural embedding inside of the trivial $\mathbb{C}^2$ bundle over $\mathbb{P}^1(\mathbb{C})$ by embeddding each line τ_p as itself in $\mathbb{C}^2$. As such, restriction of functionals yields a surjection $(\mathbb{C}^2)^* \times \mathbb{P}^1(\mathbb{C}) \to \tau^*$. From a functional $\lambda : \mathbb{C}^2 \to \mathbb{C}$, we yield a section $\tilde{\lambda}$ of τ^*. For non-zero λ, $\ker \lambda$ is one dimensional and hence $\tilde{\lambda}^{-1}(0)$ is a single point. Without loss of generality take $\lambda = e^1$, the projection onto the first factor. With respect to the local trivialization $U_2 = \mathbb{C}^2 \backslash \{z_2 = 0\}$, $\varphi : \mathbb{C} \times U_2 \to \tau$ by $(\alpha, [z_1 : z_2]) \mapsto (\alpha \frac{z_1}{z_2}, \alpha)$. We see that $\lambda \circ \varphi(\alpha, [z_1 : z_2]) = \alpha \frac{z_1}{z_2}$ which vanishes to first order at $[0 : 1]$. This tells us that $c_1(\tau^*) = 1$. This equation really means that $c_1(\tau^*) = [\mathbb{P}^1(\mathbb{C})]$.

Additivity tells us that as $\tau \otimes \tau^* \cong \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$ we have $c_1(\tau) = -1$.

4.3 Axiomatics of the Chern class

Now, that we have conceived of these two characteristic classes for complex vector bundles, the Euler class $e(E) \in H^{2\mathrm{rk}(E)}(B, \mathbb{Z})$, and the first Chern class $c_1(E) \in H^2(B, \mathbb{Z})$, one might wonder if there are other classes occupying $H^{2j}(B, \mathbb{Z})$. These classes exist, and are Chern classes. There are many ways of defining them, and we will start off with the axiomatic approach.

We will package the Chern classes together in the total chern class $c(E) \in H^*(B, \mathbb{Z})$, which lives inside the cohomology ring of B. This allows us to neatly package the generlization of property (a) from our axiomatization of the Euler class.

Definition 4.6. Let $c : \mathrm{Vect}_{\mathbb{C}} \to H^*(-, \mathbb{Z})$ be a natural transformation (i.e. an assignment of cohomology classes to each isomorphism class of vector bundles which commutes with pullbacks). Let $c_k(-)$ denote the component of $c(-)$ lying in $H^k(-,\mathbb{Z})$. c is a total Chern class if

a) $c(E \oplus E') = c(E) \cup c(E')$,

b) $c_k(\bar{E}) = (-1)^k c_k(E)$,

c) and $c_1(\tau) = -[\mathbb{P}^1(\mathbb{C})]$.

Theorem 4.7. *The total Chern class exists and is unique.*

One way to prove the existence and uniqueness of the total Chern class is through classifying spaces. We previously stated that the functor $\mathrm{Vect}^k_{\mathbb{K}} : \mathrm{Top} \to \mathrm{Set}$ is representable in the topological category, at the level of homotopy. More specifically there is a natural isomorphism between the functor $X \mapsto [X, Y]$ and $\mathrm{Vect}^k_{\mathbb{K}}$, for some space Y, with the correspondence gotten by mapping $f \in [X, Y]$ to f^*E for some bundle $E \to Y$. Since a characteristic class is a natural transformation between $\mathrm{Vect}^k_{\mathbb{K}}$ and $H^*(-,\mathbb{Z})$, the Yoneda lemma tells us that set of characteristic classes of rank k vector bundles is given by $H^*(Y,\mathbb{Z})$. Given a particular element $c \in H^*(Y,\mathbb{Z})$, we can verify that it satisfies the axioms of Definition 4.6 by understanding explicitly the properties of the chosen class. We will go down a different path, a la Grothendieck, which exhibits the Chern classes as measures of a particular bundle $X \to B$ to have cohomology isomorphic (as a ring) to $H^*(B,\mathbb{Z}) \otimes H^*(F,\mathbb{Z})$ for F the fiber of X.

4.4 The Splitting Principle

The aim of this problem session is a loosely guided proof of Theorem 4.7. Much of the coverage is inspired by the approach of [1]. There are several different proofs, with varying levels of concreteness and constructiveness. Our proof will focus on the method of the splitting principle, a method which generalizes in a particularly nice way to the algebraogeometric setting. The basic idea is as follows:

i) Given a complex vector bundle $p : E \to B$ construct a space (in reality a fiber bundle) $\pi : X_E \to B$ such that $\pi^*E \cong \bigoplus_{i \leq \mathrm{rk}(E)} L_i$ where each L_i

is a line bundle over X. These line bundles are called the Chern roots of E, as we have factored π^*E into a sum of lines.

ii) We then wish to prove that the induced map on cohomology $\pi^* : H^*(B, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is injective.

iii) Then we can produce the Chern class by forcing the product formula to hold, i.e. define $c(E)$ so that $\pi^*c(E) = \prod_i (1 + c_1(L_i))$.

iv) Then prove well definedness.

Before we delve into the splitting principle, we need to recall the structure of the cohomology of $\mathbb{P}(\mathbb{C}^k)$. Given $V \to \mathbb{C}^k$ a $k-1$ dimensional subspace, the image of $V\backslash\{0\}$ under the quotient map $\mathbb{C}^k\backslash\{0\} \to \mathbb{P}(\mathbb{C}^k)$, denoted by $\mathbb{P}(V)$ is homeomorphic to a copy of $\mathbb{P}(\mathbb{C}^{k-1})$. The complement $\mathbb{P}(V)^c$ is homeomorphic to $\mathbb{C}^{k-1}$. Thinking of $\mathbb{P}(C^2)$ as the one point compactification of $\mathbb{C}$, i.e. the CW complex obtained by attaching a 2-cell to $*$, we can use this process to inductively put a cell structure on $\mathbb{P}(\mathbb{C}^k)$, by attaching a $2k-2$ cell to $\mathbb{P}(\mathbb{C}^{k-1})$. As such the cellular homology cochain complex for $\mathbb{P}(\mathbb{C}^k)$ is

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \ldots \to 0 \to \mathbb{Z} \to 0$$

where the non-zero groups are in grading $2i$ for $k-1 \geq i \geq 0$. Since the differentials are all 0, we can write the cohomology of $\mathbb{P}(\mathbb{C}^k)$ as $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \cong \mathbb{Z}[c_1]/c_1^k$ where $c_1 = c_1(\tau)$ is placed as the generator in degree 2. In fact, this isomorphism respects the ring structure.

Proposition 4.8. *Let $p : E \to B$ be a complex vector bundle. There exists $\pi : S \to B$ such that $\pi^*E \cong \bigoplus_i L_i$ and π^* is injective on cohomology.*

Proof. Let $\pi : E \to B$ be a complex vector bundle. Define $\pi_1 : \mathbb{P}(E) \to B$ to be the fiber bundle with $\mathbb{P}(E)_p = \mathbb{P}(E_p)$. We can see that this is a fiber bundle over B by projectivizing the transition functions coming from a system of local trivializations of $E \to B$. Each of the fibers of $\mathbb{P}(E)$ carry the tautological line bundle $\tau(E_p) \to \mathbb{P}(E_p)$ and the these glue together to give a line bundle $\tau(E) \to \mathbb{P}(E)$. This yields the following short exact sequence of vector bundles

$$0 \to \tau(E) \to \pi_1^*E \to \pi_1^*E/\tau(E) \to 0.$$

Introducing a Hermitian metric allows us to split this short exact sequence (WARNING: in the algebraic category not all short exact sequences of vector bundles split, and in fact a weaker version of axiom a is used, that any short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ has $c(E) = c(E') \cup c(E''))$

Splitting this sequence allows us to write $\pi_1^* E \cong \tau(E) \oplus \pi_1^* E/\tau(E)$, now having split $\pi_1^* E$ into a sum of $\tau(E)$ and $\pi_1^* E/\tau(E)$ we can apply the process inductively to produce $X_E \to B$ for which $\pi^* E \cong \bigoplus L_i$.

Exercise 11. Describe the fiber of this repeated process of projectivization as a quotient of some Lie group by one of its subgroups.

In order to show that the projection is injective on cohomology we may apply the following theorem:

Theorem 4.9. *(Leray-Hirsch) Let* $X \to B$ *be a fiber bundle with fiber* $i : F \to X$ *such that* $H^*(F, \mathbb{Z})$ *is a free abelian group and* $i^* : H^*(X, \mathbb{Z}) \to H^*(F, \mathbb{Z})$ *is surjective. Then there exists an isomorphism* $L : H^*(B, \mathbb{Z}) \otimes H^*(F, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ *such that the following diagram commutes*

$$\begin{array}{ccc} H^*(B,\mathbb{Z}) \otimes H^*(F,\mathbb{Z}) & \xrightarrow{L} & H^*(X,\mathbb{Z}) \\ \uparrow f & \nearrow \pi^* & \\ H^*(B,\mathbb{Z}) & & \end{array}$$

where $f(b) = b \otimes 1$*.*

So, if we can prove the hypotheses of Theorem 4.9 then the injectivity of f implies the injectivity of π^*. Proving these hypotheses can be rather tedious and uninstructive, so for the sake of brevity we will work in the case where a system of local trivializations has only two elements $U, V \subset B$.

Proposition 4.10. *The cohomology of* F, $H^*(F, \mathbb{Z})$ *is a free abelian group.*

Proof. Since each level is gotten by projectivizing the quotient of the previous bundle, it suffices to prove the following:

Lemma 4.11. *Let* $E \to B$ *be a complex vector bundle. Then the cohomology of* $\mathbb{P}(E)$ *is a freely generated module over* $H^*(B, \mathbb{Z})$*, generated by* $1, x, x^2, \ldots, x^{\mathrm{rk}E-1}$ *where* $x \in H^2(\mathbb{P}(E), \mathbb{Z})$ *with* $\iota^* x = c_1(\tau)$.

Proof. Let $\iota : \mathbb{P}(\mathbb{C}^k) \to \mathbb{P}(E)$ denote the inclusion of a fiber. Naturality of $c_1(\tau)$ tells us that since $\iota^*\tau_E = \tau$ that $\iota^* c_1(\tau_E) = c_1(\tau)$. Since $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z})$ is a ring homomorphism, and $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \cong \mathbb{Z}[c_1]/c_1^k$, ι^* is necessarily surjective and the Leray-Hirsch theorem applies. □

Definition 4.12. Since $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(B, \mathbb{Z})$, generated by $1, x, x^2, \ldots, x^{\mathrm{rk}E-1}$ The equation

$$x^k = \sum_{i=1}^{k-1} a_i x^i$$

has a unique solution for some $a_i = \pi^* b_i$. We define the ith Chern class as $c_i(E) = -b_{k-i}$.

We can think of this idea as telling us that the cohomology of $\mathbb{P}(E)$ is additively $H^*(M, \mathbb{Z}) \otimes H^*(\mathbb{P}(\mathbb{C}^k))$, and the Chern classes measure the overall twisting of the ring structure on the cohomology. This gives a presentation of $H^*(\mathbb{P}(E), \mathbb{Z}) \cong H^*(B, \mathbb{Z})[x]/(\sum_{i=1}^{\mathrm{rk}E} c_i(E) x^{\mathrm{rk}E-i})$. □

Exercise 12. Using the previous ideas, write a presentation for the ring structure cohomology of $\mathbb{P}(E/\tau)$ as a quotient of $H^*(B, \mathbb{Z})[x_1, x_2]$.

Exercise 13. Set $B = U \cup V$ and $p^{-1}(U) \cong F \times U, p^{-1}(V) \cong F \times V$. Using the Mayer Vietoris sequence for $E = p^{-1}(U) \cup p^{-1}(V)$, and the Künneth formula, prove that for $x \in U \cap V$, $i^* : H^*(X, \mathbb{Z}) \to H^*(F, \mathbb{Z})$ is surjective.

□

Proceeding by induction, the preceding work tells us that as a ring $H^*(X, \mathbb{Z}) \cong H^*(B, \mathbb{Z})[x_1, \ldots, x_{\mathrm{rk}E}]/I$ for some ideal $I \subset H^*(X, \mathbb{Z})[x_1, \ldots, x_{\mathrm{rk}E}]$. We have a map $\Lambda_j : X \to \mathbb{P}(E)$ which sends the point $L_1 \oplus \ldots L_k$ to $[L_j]$. Since $\Lambda_j^*(\tau(E))a = L_j$, $\Lambda_j^* x = c_1(L_j) = x_j$ and hence $c_1(L_j)$ satisfies

$$\sum_{i=1}^{k} x_j^i c_{\mathrm{rk}E-i}(E) = 0$$

for every j. The theory of symmetric polynomials tells us that

$$c_i(E) = (-1)^i \sigma_i(x_1, \ldots, x_{\mathrm{rk}E})$$

for σ_i the ith elementary symmetric polynomial

$$\sigma_i(x_1, \ldots, x_{\mathrm{rk}E}) = \sum_{I \subset [\mathrm{rk}E], |I| = i} x^I$$

so that $\sum c_i(E) = \sum \sigma_i(x_1, \ldots, x_{\mathrm{rk}E}) = \prod_{j=1}^{k}(1 + x_j) = \prod_{j=1}^{k}(1 + c_1(L_j))$ so that our definitions coincide.

Remark. When defining the Chern classes, we never proved that our definition did not depend on the choice of lifts of the generators of $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z})$, but since we have exhibited the total Chern as a polynomial in $(1 + c_1(L_j))$, it does not depend on the choice of lifts.

Exercise 14. Prove the product formula $c(E \oplus E') = c(E)c(E')$ using the splitting principle.

Exercise 15. For L a complex line bundle and E rank 2 complex vector bundle, find a formula for $c(L \otimes E')$.

Exercise 16. Prove that $c(E)$ is natural.

Hint: Note that the since the construction of X_E really comes from gluing together X_{E_b}, that the construction commutes with pullbacks, i.e. $f^*(X_E) = X_{f^*E}$, and the induced map $\tilde{f} : X_{f^*E} \to X_E$ has $\tilde{f}^* L_j = \tilde{L}_j$, where $L_j \to X_E$ and $\tilde{L}_j \to X_{f^*E}$ are the Chern roots.

5 Principal Bundles, Connections, and Curvature

5.1 Principal Bundles

The principal idea behind principal bundles is to understand families by understanding their automorphisms, a la Felix Klein. This is best manifest in the correspondence between vector bundles and $GL_n(\mathbb{K})$ bundles. Given $E \to B$ a rank k $\mathbb{K}$-vector bundle over B, we can associate a bundle of frames $FE \to B$, the fiber FE_p will be the space of bases of E_p. Formally this can be thought of as isomorphisms from $\mathbb{K}^n \to E_p$. Hence $FE \subset \mathrm{Hom}(\mathbb{K}^n \times B, E)$. The natural matrix multiplication of $GL_n(\mathbb{K})$ on $\mathrm{Hom}(\mathbb{K}^n, E)$ restricts to FE and $FE/GL_n(\mathbb{K}) \cong B$. This action is a right action. This space FE is called a principal $GL_n(\mathbb{K})$ bundle

Definition 5.1. Let G be a topological group. A triple $\pi : P \to B$, where P is a space with right action by G, is called a principal G bundle if π is G invariant, $P/G \cong B$ by the orbit map, and for each $b \in B$ there exists a $U \subset B$ open and an isomorphism of G spaces

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & G \times U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

where $G \times U$ is equipped with the action $g(g', u) = (g'g, u)$.

The latter part of this definition means that G acts on each fiber of π freely and transitively, i.e. P_b is a G-torsor. We can thus distill the previous definition to: a principal G-bundle is a bundle of G-torsors.

Proposition 5.2. *Let G be the trivial G torsor. The space of automorphisms of G is given by G itself.*

Proof. If $\varphi : G \to G$ is a G-torsor automorphism, then $\varphi(g) = \varphi(1)g$ so there is a correspondence between $\mathrm{Aut}_{\mathrm{Tors}}(G)$ and G given by left multiplication by G. $\square$

This means that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} : G \times (U_\alpha \cap U_\beta) \to G \times (U_\alpha \cap U_\beta)$ must be given by a map $U_\alpha \cap U_\beta \to G$.

To each space B, we may associate the set of isomorphism classes of principal G-bundles $\mathrm{Bun}_G(B)$ (a morphism of principal G bundles is a G equivariant map $P \to P'$). $\mathrm{Bun}_G(-)$ defines a contravariant functor $\mathrm{Top} \to \mathrm{Set}$ with morphisms $\mathrm{Bun}_G(f)(P) = f^*P$ for a map $f : B' \to B$.

Proposition 5.3. *There is a natural isomorphism of functors $\mathrm{Vect}_{\mathbb{K}}^n \implies \mathrm{Bun}_{GL_n(\mathbb{K})}$, taking a rank n vector bundle over B to its bundle of frames.*

Proof. As we have already defined such a natural transformation, we need only prove that it has an inverse. Given a principal $GL_n(\mathbb{K})$ bundle $P \to B$, associate to it the vector bundle $(P \times \mathbb{K}^n)/GL_n(\mathbb{K})$ equipped with the right action $g(p, v) = (gp, g^{-1}v)$. Given a local trivialization for P, $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G \times U$, this induces an equivariant identification $\pi^{-1}(U_\alpha) \times \mathbb{K}^n \to G \times U_\alpha \times \mathbb{K}^n$ and hence a homeomorphism $(\pi^{-1}(U_\alpha) \times \mathbb{K}^n)/GL_n(\mathbb{K}) \to U_\alpha \times \mathbb{K}^n$. On overlaps, the map $G \times (U_\alpha \cap U_\beta) \times \mathbb{K}^n \to G \times (U_\alpha \cap U_\beta) \times \mathbb{K}^n$ will be given by $(g, u, v) \mapsto (\varphi_\alpha \circ \varphi_\beta^{-1}(g), u, v) \sim (g, u, \varphi_\beta \circ \varphi_\alpha^{-1}(v))$ so that the transition maps induced by these trivializations are still valued in $GL_n(\mathbb{K})$ and hence $(P \times \mathbb{K}^n)/GL_n(\mathbb{K})$ is a vector bundle.

There is an isomorphism between $(FE \times \mathbb{K}^n)/GL_n(\mathbb{K})$ and E given by the map $[p, v] \mapsto p(v)$. Matrix multiplication means that this map is well defined, as $[pg, g^{-1}v] \mapsto p(gg^{-1}v) = p(v)$ and since every element of E_b may be written uniquely as $v_1 e_1 + \ldots v_n e_n$ for some vectors $e_1, \ldots, e_n$ this map is bijective. Since this map is given by applying p, it is linear and hence a vector bundle isomorphism. This tells us that instead of dealing with vector bundles, we could deal with principal $GL_n(\mathbb{K})$-bundles. $\square$

The process by which we take $P \times \mathbb{K}^n$ and mod out by the antidiagonal action is called the associated bundle construction. Generally, given a left representation of $\rho : G \to \mathrm{Homeo}(F)$ we can form the associated bundle to P with fiber F, by $P \times_\rho F := (P \times F)/G$ where the G action on $P \times F$ is $g(p, f) = (gp, \rho(g^{-1})f)$. This is always a fiber bundle with fiber F and transition maps valued in $\rho(G)$.

Example 7. The associated bundle construction allows us to obtain all of the bundles related to a given principal G-bundle by studying its representation theory. Given $\pi : E \to B$ a vector bundle and $FE \to B$ its bundle of frames, we can form the dual vector bundle $E^* = \mathrm{Hom}(E, \mathbb{K})$ by taking the bundle associated to the dual representation of $GL_n(\mathbb{K})$. More concretely, the action of $GL_n(\mathbb{K})$ on $\mathbb{K}^n$, $(g, v) \mapsto gv$ dualizes, to yield an action of $GL_n(\mathbb{K})$ on $(\mathbb{K}^n)^*$ by $(g, \alpha) \mapsto (v \mapsto \alpha(gv))$. Since dualization is contravariant, this would naturally lead to a right action, we will take the inverse in order to get the corresponding left action. Denote the standard representation of

$GL_n(\mathbb{K})$ and the dual representation by ρ and ρ^* respectively. Then define $\tilde{E}^* = FE \times_{\rho^*} (\mathbb{K}^n)^*$, this is meant to be the dual bundle to E. We can see that as presented there is a natural evaluation map $E \times_\pi \tilde{E}^* \to \mathbb{K} \times B$ as follows. The standard evaluation map $K^n \times (\mathbb{K}^n)^* \to \mathbb{K}$ is invariant under the product action of $GL_n(\mathbb{K})$ as $g \cdot (u, \lambda) = (gu, (g^{-1})^*\lambda) \mapsto \lambda(g^{-1}gu) = \lambda(u)$ for all $(u, \lambda) \in \mathbb{K}^n \times (\mathbb{K}^n)^*$. Therefore, we have an equivariant map $FE \times \mathbb{K}^n \times (\mathbb{K}^n)^* \to FE \times \mathbb{K}$, where $\mathbb{K}^n \times (\mathbb{K}^n)^*$ is given the right action induced by the left action above and $\mathbb{K}$ has the trivial action. This equivariant map then descends to $FE \times_{\rho \times \rho^*} (\mathbb{K}^n \times (\mathbb{K}^n)^*) \to B \times \mathbb{K}$. The map $[(p, u, \lambda)] \mapsto ([p, u], [p, \lambda])$, $\Psi : FE \times_{\rho \times \rho^*} (K^n \times (\mathbb{K}^n)^*) \to (FE \times_\rho \mathbb{K}^n) \times_\pi (FE \times_{\rho^*} (\mathbb{K}^n)^*) \cong E \times_\pi \tilde{E}^*$ is a bundle isomorphism, so that we get a well defined, fiberwise bilinear map

$$(FE \times_\rho \mathbb{K}^n) \times_\pi (FE \times_{\rho^*} (\mathbb{K}^n)^*) \to \mathbb{K} \times B$$

Exercise 17. Prove that the map Ψ above is an isomorphism, and the induced map $\tilde{E}^* \to \mathrm{Hom}(E, \mathbb{K})$ is as well.

Remark. The associated bundle $P \times_\rho F$ should not be confused with a fibered product. The notation $P \times_\rho F$ is less than optimal, but it is rather compact.

While vector bundles have many sections, a general principal bundle has no global sections. In fact, a principal G bundle is trivial if and only if it has a global section. Given a global section $\sigma : B \to P$, there is an isomorphism $G \times B \to P$ given by $(g, b) \mapsto \sigma(b)g$.

Remark. While a vector bundle without structure produces a $GL_n(\mathbb{K})$ principal bundle over its base, additional structure allows us to yield principal H bundles for H the fiberwise automorphism group of the structure.

Example 8. If $E \to B$ is an $\mathbb{R}$-vector bundle of rank n, and is equipped with an orientation, we may construct $FE^{or} \to B$ the bundle of oriented frames of E. We have $FE^{or} \subset FE \subset \mathrm{Hom}(\mathbb{R}^n \times B, E)$ where a map $T : \mathbb{R}^n \to E_b$ is orientation preserving if the orientation $(T^{-1}(u_1), \dots, T^{-1}(u_n))$ agrees with the standard orientation on $\mathbb{R}^n$ for $(u_1, \dots, u_n)$ an oriented bases of E_b. Since $GL_n(\mathbb{R})^+$ preserves the standard orientation on $\mathbb{R}^n$, $FE^{or} \to B$ is a principal $GL_n(\mathbb{R})^+$ bundle.

Example 9. If $E \to B$ is a rank n $\mathbb{R}$-vector bundle equipped with a Euclidean structure $\langle\cdot,\cdot\rangle_E$, we may construct $OFE \to B$, as $OFE \subset FE \subset \mathrm{Hom}(\mathbb{R}^n \times B)$ the bundle of orthonormal frames of E, where a frame $T : \mathbb{K}^n \to E_b$ is orthonormal if $\langle T_b(u), T_b(v)\rangle_{E_b} = \langle u, v\rangle_{\mathbb{R}^n}$ for all $u, v \in \mathbb{R}^n$ and $b \in B$. In other words, OFE_b is the space of isometric isomorphisms between $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)_{\mathbb{R}^n}$ and $(E_b, \langle\cdot,\cdot\rangle_{E_b})$. The action of $O(n)$ (isometric automorphisms of $\mathbb{R}^n$) yields an action on OFE making OFE a principal $O(n)$- bundle.

Example 10. If we consider $E \to B$ a rank n $\mathbb{C}$-vector bundle equipped with a Hermitian structure, similar arguments as above yield $UFE \to B$, the space of unitary frames of E. More specifically, this is done by looking at isomorphisms from $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ and $(E_b, \langle\cdot,\cdot\rangle_{E_b})$. The unitary frame bundle of E, $UFE \to B$ is a principal $U(n)$-bundle.

Example 11. From each of the preceding examples, we can recover E by the mixing construction. If $H \subset GL_n(\mathbb{K})$ is the relevant automorphism group, and we denote $\iota : H \to GL_n(\mathbb{K})$ the standard linear representation of H, then the argument from the proof of Proposition 5.3 gives an isomorphism between $P \times_\iota \mathbb{K}^n \cong E$. It is not unreasonable then to say we have the following natural isomorphisms of functors:

$$\begin{aligned} \mathrm{Vect}_{\mathbb{R}}^{n,or} &\Longrightarrow \mathrm{Bun}_{GL_n(\mathbb{R})^+} \\ \mathrm{Vect}^{n,eucl} &\Longrightarrow \mathrm{Bun}_{O(n)} \\ \mathrm{Vect}^{n,herm} &\Longrightarrow \mathrm{Bun}_{U(n)} \end{aligned}$$

For the rest of this course, we will focus on the differential geometry and topology of smooth principal G-bundles for G a Lie group. To start, we will want to study smooth manifolds with right G actions.

Definition 5.4. Let M be a space equipped with right G action (a right G-space for short). There is a Lie algebra anti-homomorphism $\mathfrak{g} \to \mathfrak{X}(M), \xi \mapsto X_\xi$ called the fundamental vector field map (anti-homomorphism simply means $[X_\xi, X_\eta] = -X_{[\xi,\eta]}$) defined by $X_\xi(p) = \frac{d}{dt}\big|_{t=0} \exp(t\xi) \cdot p$. These vector fields are fundamental because they locally generate the action of G.

The evaluation map $(\xi, p) \mapsto X_\xi(p)$ gives a vector bundle morphism $\mathfrak{g} \times M \to TM$

Proposition 5.5. *Let* M *be a right* G*-space. If the* G *action on* M *is free then the evaluation map* $\mathfrak{g} \times M \to TM$ *is injective. Moreover, this map is equivariant with respect to the tangent lift of the* G *action on* TM *and the* G *action on* $\mathfrak{g} \times M$, $(\xi, p) \mapsto (\mathrm{Ad}_{g^{-1}}\xi, gp)$

Proof. Assume that $\mathfrak{g} \times M \to TM$ is not injective. That means there is some (ξ, p) for which $\frac{d}{dt}\big|_{t=0} \exp(t\xi)p = 0$. Since X_ξ is a time independent vector field this means that $\exp(t\xi)p = p$ for all t for which $\exp(t\xi)$ is defined, contradicting freeness.

To see that this map intertwines the desired actions, it is helpful to note that $g(\exp(t\xi)p) = (\exp(t\xi)g)p$ so

$$\begin{aligned} g(X_\xi(p)) = \frac{d}{dt}\bigg|_{t=0} (\exp(t\xi)g)p &= \frac{d}{dt}\bigg|_{t=0} (gg^{-1}\exp(t\xi)g)p \\ &= \frac{d}{dt}\bigg|_{t=0} (g^{-1}\exp(t\xi)g)gp \\ &= X_{\mathrm{Ad}_{g^{-1}}\xi}(gp) \end{aligned}$$

□

The preceding proposition only requires that the action of G is locally free, i.e. a neighborhood of the identity acts freely, but we only care about the case in which this action is free. Since a relative open neighborhood of an orbit of G is the image of $\exp(-)p : G \to M$, this tells us that the image of $\mathfrak{g}$ in T_pM is the tangent space to the orbit of p.

5.2 Connections on Principal Bundles

Let $\pi : P \to B$ be a principal G bundle. Our local trivializations ensure that G acts freely on P and hence $\mathfrak{g} \times P$ naturally lives as a subbundle of TP. We actually have more than this, there is a short exact sequence of vector bundles (meaning that it is a pair of vector bundle morphisms which are short exact sequences on each of the fibers.)

$$0 \to \mathfrak{g} \times P \to TP \xrightarrow{d\pi} \pi^*TB \to 0$$

The bundle $\mathfrak{g} \times P$ is called the vertical bundle, while π^*TB is the horizontal bundle. The standard picture of a principal bundle displays the fibers of π vertically and local sections horizontally.

The idea of a connection is to allow us to do the process of parallel transport, that is given a path $\gamma : I \to B$ and a choice of lift of $\gamma(0)$ produce a lift $\hat{\gamma} : I \to P$. By this process, we can transport (parallely even) data at $\pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$.

Definition 5.6. A(n Ehresmann) connection on a principal G-bundle is a choice of G equivariant splitting of the preceding short exact sequence. We can think of this as a map bundle map $\omega : TP \to \mathfrak{g} \times P$ which has $\omega(X_\xi) = \xi$ and $\omega(gv) = \mathrm{Ad}_{g^{-1}}(\omega(v))$.

The geometric interpretation of this is as a way to write $TP \cong \mathfrak{g} \oplus \pi^*TB, v \mapsto (\omega(v), d\pi(v))$. A vector field Y on B induces a section of $\tilde{Y}$ of π^*TB by $\tilde{Y}(p) = Y(\pi(p))$. The image of π^*TB under the splitting, the Horizontal distribution, will be denoted as H. Denote the composition $TP \to \pi^*TB \to TP$ by h.The splitting allows us to realize $\tilde{Y}$ as a vector field $\hat{Y}$ on P which is π-related to Y (i.e. $d\pi(\hat{Y}(p)) = Y(\pi(b))$) so that the flow of $\hat{Y}$ (if it exists) maps down to the flow of Y. The vector field $\hat{Y}$ is called the horizontal lift of Y as it is the unique horizontal vector field which is π related to Y. Since ω is given by a map to the trivial vector bundle $\mathfrak{g} \times P$, we can recast it as a $\mathfrak{g}$-valued 1-form on P, i.e. $\omega \in \Omega^1(P, \mathfrak{g})$.

Example 12. A Riemannian metric g on a smooth manifold M is equivalently a choice of smooth Euclidean structure on $TM \to M$. As such, to any Riemannian manifold there is an associated $O(n)$ bundle of orthonormal frames $OFM := OFTM \to M$. The content of the fundamental theorem of Riemannian geometry is that OFM admits a canonical choice of connection with vanishing torsion. From this connection, we can specify a covariant derivative using the yet-to-be-defined exterior covariant derivative.

Remark. We can extend the exterior derivative to any trivial vector bundle in a natural way: Given $\alpha \in \Omega^k(P, V)$, choosing an isomorphism $\varphi : V \to \mathbb{R}^n$ means we can realize α as an element α' of $\Omega^k(P, \mathbb{R}^n)$ or as an n-tuple of forms $\alpha'_i := ((v_1, \ldots, v_k) \mapsto \Phi(\alpha(v_1, \ldots, v_k))_i)$ then $d\alpha := \Phi^{-1}(d\alpha')$.

Exercise 18. Show that the extension of d to $\Omega^k(P, V)$ doesn't depend on our choice of Φ. Show that as a module over $\Omega^*(P)$, d defines a (super)-derivation of degree 1 on $\Omega^*(P, V)$.

Proposition 5.7. *Let $P \to B$ be a principal G-bundle. There exists a connection on P.*

Proof. Let $(U_\alpha, \varphi_\alpha)$ be a system of local trivializations of P, with $\psi_\alpha : B \to \mathbb{R}$ a partition of unity with supp $(\psi_\alpha) \subset U_\alpha$. Define $\omega_\alpha \in \Omega^1(\pi^{-1}(U_\alpha), \mathfrak{g})$ by $\omega_\alpha := \varphi_\alpha^* \tilde{\omega}_\alpha$ where $\tilde{\omega}_\alpha \in \Omega^1(G \times U_\alpha, \mathfrak{g})$ is given by the differential of the projection $G \times U_\alpha \to G$.The form ω_α is a connection form on $\pi^{-1}(U_\alpha)$ since φ_α is G equivariant. Define $\omega = \sum \pi^* \psi_\alpha \omega_\alpha$. Then

$$\begin{aligned} \omega(X_\xi(p)) &= \sum \psi_\alpha(\pi(p))\omega_\alpha(X_\xi) \\ &= \Big(\sum \psi_\alpha(\pi(p))\Big)\xi = \xi \end{aligned}$$

And

$$\begin{aligned} \omega(gv) &= \sum \psi_\alpha(\pi(gp))\omega_\alpha(gv) \\ &= \sum \psi_\alpha(\pi(p))\mathrm{Ad}_{g^{-1}}\omega_\alpha(v) \\ &= \mathrm{Ad}_{g^{-1}}\Big(\sum \psi_\alpha(\pi(p))\omega_\alpha(v)\Big) \\ &= \mathrm{Ad}_{g^{-1}}\omega(v) \end{aligned}$$

□

Remark. The structure of the space of connections is rather simple. Given two connections ω, ω', their difference $\omega - \omega'$ is an equivariant 1-form which vanishes on the vertical distribution $\mathfrak{g} \times P$.

Given a connection ω on a principal bundle $P \to B$, we can try to implement parallel transport. Take $\gamma : I \to B$. We can pull P back by this map to yield $\gamma^* P \to I$. The pullback map yields $\gamma^* P \to P$ which we can pull ω back by to yield a connection form $\gamma^*\omega$ on $\gamma^* P$. Let X be the horizontal lift of the translational vector field $\frac{\partial}{\partial t}$. Equivariance of ω means that X is G invariant.

Proposition 5.8. *The flow of X is defined for times $[0, 1]$ and hence induces a G equivariant map $P_{\gamma(0)} \to P_{\gamma(1)}$*

Proof. Given $p \in P_{\gamma(0)}$, the theory of ODE's guarantees the existence of an integral curve for X along p for times $[0, \epsilon)$. If $\hat{\gamma}$ is such an integral curve, then $g\hat{\gamma}$ is the integral curve corresponding to gp since $\frac{d}{dt}g\hat{\gamma}(t) = g\frac{d}{dt}\hat{\gamma}(t) = gX(\gamma(t))$ so that the integral curve through gp exists for times $(-\epsilon, \epsilon)$ (choosing a suitable extension of $\gamma : (-\delta, 1+\delta))$, $\epsilon < 1$. This means that the local flow yields a map $P_{\gamma(0)} \to P_{\gamma(\epsilon')}$ for $\epsilon' < \epsilon$. There is a continuous map $I \to [0, \infty)$ taking

$$s \mapsto \sup_t \{ \text{ the flow of } X \text{ through } P_{\gamma(s)} \text{ is defined for times } (-t, t)\}.$$

Since I is compact, and the flows all exist for some time, the image of this map does not contain 0 and by a variation of the uniform time lemma [7] Lemma 9.15, X has a flow for times $[0, 1]$.

G equivariance is manifest as X is invariant. □

Now, given a loop $\gamma : I \to B$, starting and ending at b we have a map $P_b \to P_b$ which must be given by multiplication by some element of G. It is sensible to ask, how does this map depend on our choice of loop? In particular, if $\gamma : I \to B$ is a contractible loop, is $P_b \to P_b$ necessarily the identity? Assuming γ is an embedding, we can instead choose to deal with global vector fields on B and their horizontal lifts. One could compute the parallel transport by extending $\frac{\partial}{\partial t}$ to a vector field on B and flowing its horizontal lift. A new way to phrase our original question is, given X and Y commuting vector fields, do the horizontal lifts of X and Y, $\hat{X}$ and $\hat{Y}$ commute? or is the map $\mathfrak{X}(B) \to \mathfrak{X}(P)$ a Lie algebra homomorphism? The failure for the horizontal lift map to be a Lie algebra homomorphism is naturally encoded in $\Omega'(X, Y) := \frac{1}{2}(([X, Y] - X_{\omega([X,Y])} - [X, Y]) = -\frac{1}{2}X_{\omega([X,Y])}$ since $X \mapsto \hat{X}$ is a Lie algebra homomorphism if and only if ω is identically 0.

Theorem 5.9. *(Structure Equation) Let $\Omega \in \Omega^2(P, \mathfrak{g})$ defined by*

$$\Omega(A, B) = d\omega(A, B) + [\omega(A), \omega(B)] = (d\omega + \frac{1}{2}[\omega, \omega])(A, B).$$

Then for any horizontal vector fields X and Y, we have

$$\Omega(X, Y) = -\frac{1}{2}\omega([X, Y]) = \omega(\Omega'(X, Y)).$$

Proof. Since X and Y are horizontal,

$$\begin{aligned}\Omega(X,Y) &= d\omega(X,Y) + [\omega(X),\omega(Y)] \\ &= \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X,Y])) + 0 \\ &= -\frac{1}{2}\omega([X,Y]) \\ &= \omega(\Omega'(X,Y))\end{aligned}$$

□

We can view a connection as a sort of universal covariant derivative on vector bundles associated to $P \to B$ in the following way. Given a linear representation $\rho : G \to GL(V)$, the associated bundle construction gives rise to an equivalence between sections of $E = P \times_\rho V \to B$ and equivariant functions $P \to V$, and E-valued forms with equivariant V-valued forms which vanish on the vertical distribution. Then, given an E-valued k-form η, with corresponding form on P denoted $\tilde{\eta}$, we can take its exterior derivative $d\tilde{\eta}$. Using our connection $D\tilde{\eta} = d\tilde{\eta} \circ h$ is an equivariant V-valued $k+1$ form on P which vanishes on the vertical distribution and hence descends to a unique E-valued $k+1$ form $d_\omega\eta$.

Exercise 19. Prove that $d_\omega : \Omega^*(B,E) \to \Omega^*(B,E)$ is a degree $+1$ derivation (viewing $\Omega^*(B,E)$ as a module over $\Omega^*(B)$).

Because $d\omega$ is equivariant, Ω pushes down to a $P \times_{Ad} \mathfrak{g}$ valued 2-form on B and hence we can take its covariant derivative. The following result tells us that Ω is covariantly constant

Proposition 5.10. *(Bianchi's identity) The covariant exterior derivative of* Ω *vanishes, i.e.* $d_\omega\Omega = 0$

Proof. Since $D\Omega$ only depends on its values on H, we simply need to show that $d_\omega\Omega_p(u,v,w) = 0$ for all $u,v,w \in T_{\pi(p)}B$. Let X, Y, Z be extensions of

u, v, w to horizontal vector fields on P. We then have:

$$\begin{aligned}(D\Omega)_p(u, v, w) &= d\left(d\omega + \frac{1}{2}[\omega, \omega]\right)(X, Y, Z)|_p \\ &= \frac{1}{2}(d[\omega, \omega])(X, Y, Z)|_p \\ &= X[\omega(Y), \omega(Z)] - Y[\omega(X), \omega(Z)] + Z[\omega(X), \omega(Y)] \\ &\quad - [\omega([X, Y]), \omega(Z)] + [\omega([X, Z]), \omega(Y)] - [\omega([Y, Z]), \omega(X)]\end{aligned}$$

Since $\omega(X) = \omega(Y) = \omega(Z) = 0$ we see that each term in the preceding expression vanishes identically. □

5.3 Chern-Weil Theory

Let $f : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ be an adjoint invariant symmetric k-multilinear map. Adjoint invariance simply means $Ad_g^* f = f$. Given an equivariant $\mathfrak{g}$ valued l-form η on P which vanishes on the vertical distribution, $f(\eta, \eta, \ldots, \eta)$ gives an invariant element of $\Omega^{kl}(P, \mathbb{R})$ which vanishes on the vertical distribution. As such it descends to a unique form $\beta \in \Omega^{kl}(B, \mathbb{R})$ with $\pi^*\beta = f(\eta, \ldots, \eta)$. We can apply this process to the curvature form of a connection Ω in order to yield a well defined form $f(\bar{\Omega}) \in \Omega^{2k}(B, \mathbb{R})$. The following proposition is what allows us to obtain well defined cohomology classes:

Proposition 5.11. *Let ω be a connection on $\pi : P \to B$ and f an adjoint invariant symmetric k-multilinear map. The form $f(\bar{\Omega}) \in \Omega^{2k}(B, \mathbb{R})$ is closed and its cohomology class $[f(\bar{\Omega})] \in H^{2k}(B, \mathbb{R})$ does not depend on the choice of connection.*

Proof. Since $\pi^* df(\bar{\Omega}) = df(\Omega)$, it suffices to prove that $df(\Omega) = 0$. Invariance and the fact that Ω vanishes on the vertical distribution mean that $df(\Omega) = Df(\Omega)$. Because $Df(\Omega)$ will be a polynomial in $f(\Omega)$ and $D\Omega$, the Bianci identity guarantees that $Df(\Omega) = 0$.

A proof that $f(\bar{\Omega})$ does not depend on a choice of connection can be found in [6] Vol II. □

We can package this process into the Chern-Weil homomorphism, $w_P : I(G) \to H^*(B, \mathbb{R})$ where $I(G)$ is the space of adjoint invariant polynomials on $\mathfrak{g}$. At this point, it is not clear that this process is related at all to the characteristic classes of the first half of the course, however, the following lemma gives some justification:

Proposition 5.12. *The homomorphism w_P is functorial with respect to pullbacks of principal bundles, i.e. $w_{F^*P} = F^* \circ w_P$ for a map $F : B' \to B$.*

Proof. The pullback diagram

$$\begin{array}{ccc} f^*P & \xrightarrow{\tilde{F}} & P \\ {\scriptstyle F^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[F]{} & B \end{array}$$

gives us a G-equivariant map $\tilde{F} : F^*P \to P$ and hence $T\tilde{F}$ takes $V_{F^*\pi}$ to V_π, so $\tilde{F}^*\omega$ yields a connection on F^*P. Then $\tilde{F}^*\Omega_\omega = \Omega_{\tilde{F}^*\omega}$ and $F^*f(\Omega) = f(\overline{\tilde{F}^*\Omega}) = f(\overline{\Omega_{\tilde{F}^*\omega}})$. □

We now turn our attention to the case of complex vector bundles. From a Hermitian line bundle $L \to B$, the bundle of frames is simply $SL = \{v \in L, \|v\|^2 = 1\} \to B$, equipped with the multiplicative $U(1)$ action. A connection on FL is simply a choice of invariant $\mathfrak{u}(1)$ valued 1-form on SL. We have $U(1)$ and hence $\mathfrak{u}(1) \cong i\mathbb{R}$. Since $U(1)$ is abelian, any polynomial is invariant. The function f which maps $i\alpha$ to $\frac{\alpha}{2\pi}$ will generate $I(U(1))$. This class will correspond to the first Chern class.

A first calculation, to prove that our theory gives results that square with our previous characteristic classes, would be a good idea now.

Proposition 5.13. *For $\tau \to \mathbb{P}(\mathbb{C}^2)$, $w_\tau(f) = -[\mathbb{P}(\mathbb{C}^2)] \in H^2(\mathbb{P}(\mathbb{C}^2), \mathbb{R})$*

Proof. By equipping $\mathbb{C}^2$ with the standard Hermitian form, and seeing that $F\tau = \mathbb{C}^2 \backslash \{0\}$, we have $S\tau = S^3$ equipped with the inverse of the standard $U(1)$ action, $(\exp(i\theta), z) \mapsto \exp(-i\theta)z$. With respect to the canonical trivialization of $T\mathbb{C}^k|_{S^3}$, the $U(1)$ action is generated by the vector field $z \mapsto -iz$ (since $\exp(-it)z$ differentiates to $-i\exp(it)z$). Because the standard inner product on $\mathbb{C}^2 \cong \mathbb{R}^4$ is given by $\mathrm{Re}(h)$ we have that $TS^3 = \ker(\langle -, z\rangle)|_{S^3}$. An Ehresmann connection on $S^3 \to S^2$, can be gotten by taking the orthogonal compliment to the vertical bundle with respect to the standard inner product, hence $H = \ker(\mathrm{Re}(h(-, iz))|_{TS^3}$. If $\sigma \to S^3$ is the subbundle of $\mathbb{C}^2 \times S^3$ written as $\{(\lambda z_1, \lambda z_2, z_1, z_2)|(z_1, z_2) \in S^3, \lambda \in \mathbb{C}\}$, then $H = \sigma^{\perp}$. We can then write our connection form as

$$\omega = \mathrm{Re}(h(-, iz)) = -i(y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2).$$

Over the chart U_1 of $\mathbb{P}(\mathbb{C}^2)$, we have a section of $S^3 \to \mathbb{P}(\mathbb{C}^2)$ given by $s : z \mapsto \left(\frac{z}{\sqrt{|z|^2+1}}, \frac{1}{\sqrt{|z|^2+1}}\right)$. Under this section we have $s^*\omega = -i\frac{ydx - xdy}{x^2+y^2+1}$ so that $\Omega = ds^*\omega = \frac{-2ir^2 dr \wedge d\theta}{(r^2+1)^2}$ in polar coordinates. Since $U_1^c = \{[1 : 0]\}$, integration over U_1 will yield the value of the whole integral, and

$$\langle c_1(\tau), [\mathbb{P}(\mathbb{C}^2)]\rangle = \int_{\mathbb{P}(\mathbb{C}^2)} \frac{1}{2\pi i}\Omega = \frac{1}{2\pi i}\int_{\mathbb{C}} -\frac{r^2}{(r^2+1)^2} dr \wedge d\theta = -1$$

as desired. □

Exercise 20. Fill in the details of the preceding calculation.

Much of the following treatment comes from [6]. In order to link Chern-Weil theory to our axiomatic treatments of the Euler and Chern classes, we first have to define these classes in our theory. For Chern classes, we will make use of the existence of a Hermitian metric, which makes every complex vector bundle associated to a principal $U(n)$ bundle. The theory of polynomials tells us that invariant symmetric k-multilinear functions $\mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$ are the same as adjoint invariant degree k polynomials on $\mathfrak{g}$. For $\lambda \in \mathbb{R}$, consider the polynomial $\det(\lambda I + iX) = \lambda^n - f_1(X)\lambda^{n-1} + f_2(X)\lambda^{n-2} + \ldots + (-1)^n f_n(X)$ for $X \in \mathfrak{u}(n)$ (does this polynomial look familiar?)

Proposition 5.14. *Each* $f_i \in I(U(n))$ *and the collection* $\{f_i\}_{i \leq n}$ *generates* $I(U(n))$.

Exercise 21. Prove that $f_i \in I(U(n))$ for $0 \leq i \leq n$.

For a proof of the second fact, see [6]

Definition 5.15. Let E be a Hermitian vector bundle and FE its corresponding bundle of unitary frames. We define the Chern-Weil theory Chern classes as $c_i^{ChWe}(E) = w_{FE}(f_i)$, and the total Chern class $c^{ChWe}(E) = w_{FE}(\sum_{i=0}^{1} f_i)$.

Remark. Let $T^n = U(1)^n$ and $\mathfrak{t}$ its Lie algebra. The inclusion $T^n \to U(n)$ as the diagonal matrices induces a map $I(U(n)) \to I(T^n)$ by pullback. The Lie algebra $\mathfrak{t}$ can be written as the diagonal imaginary matrices

$$\mathfrak{t} = \{\text{diag}(ix_1, \ldots, ix_n) | a_i \in \mathbb{R}\}$$

and hence

$$\begin{aligned}\det(\lambda I + i\text{diag}(ix_1, \ldots, ix_n)) &= \det(\lambda I - \text{diag}(x_1, \ldots, x_n)) \\ &= \det(\text{diag}(\lambda - x_1, \ldots, \lambda - x_n)) \\ &= \prod_{i=1}^{n} (\lambda - x_i).\end{aligned}$$

This means that on the restriction to T^n each f_i is given by the ith elementary symmetric polynomial in x_i.

The spectral theorem tells us that for $h \in \mathfrak{u}(n)$ is diagonalizable by a matrix in $U(n)$ so $h = UtU^{-1}$ for some $t \in T^n$ and $U \in U(n)$. As such, an element of $I(U(n))$ only depends on the eigenvalues of h and $f_i(h)$ is the ith symmetric polynomial in the eigenvalues of h.

Part of the power of Chern-Weil theory, is that it allows us to prove identities about characteristic classes by doing matrix algebra.

Proposition 5.16. *The Chern-Weil theory Chern class satisfies the product formula, i.e.* $c^{ChWe}(E \oplus E') = c^{ChWe}(E)c^{ChWe}(E')$

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