Characteristic Classes, Principal Bundles, and Curvature: Exercises

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2 Introduction to Vector Bundles

2.1 Why Vector Bundles?

Proposition 2.1. *If* $M \to N$ *is an embedded closed submanifold of middle dimension, the self intersection of* M *in* N *is equal to that of* M *in* $N_{M/N}$.

Exercise 1. Prove proposition 1.2.

2.2 Formal Definitions and Operations on Vector bundles

Exercise 2. Let $E \to B$ be a vector bundle. Prove that for each $p \in B$, $\pi^{-1}(B)$ carries a natural vector space structure, and is isomorphic to $\mathbb{K}^k$.

Example 1. The tangent bundle $TM \to M$ to a smooth manifold M^n is a prototypical example of a vector bundle.

Exercise 3. Prove, using your favorite definition of the tangent bundle, that $TM \to M$ is a vector bundle.

Example 2. Consider the trivial line bundle over $\mathbb{R}$, $\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This admits a $\mathbb{Z}$ action by $a(x, y) = ((-1)^a x, y + a)$. Denote $\rho : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $\rho' : \mathbb{R} \times \mathbb{R} \to (\mathbb{R} \times \mathbb{R})/\mathbb{Z}$ the projection maps. Since the $\mathbb{Z}$ action on $\mathbb{R} \times \mathbb{R}$ commutes with the projection $\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, there is unique map $(\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \xrightarrow{\mathrm{pr}_2} & \mathbb{R} \\ {\scriptstyle\rho'}\downarrow & & \downarrow{\scriptstyle\rho} \\ (\mathbb{R} \times \mathbb{R})/\mathbb{Z} & \xrightarrow[\pi]{} & \mathbb{R}/\mathbb{Z} \end{array}$$

The quotient $\pi : (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ admits a vector bundle structure. Thinking more geometrically, one can picture a fundamental domain of $\mathbb{R} \times \mathbb{R}$ as a strip $\mathbb{R} \times [0, 1]$ and the identification is gotten by identifying $\mathbb{R} \times 0$ with $\mathbb{R} \times 1$ via $(x, 0) \mapsto (-x, 1)$. This yields the classic picture of the Möbius strip!

Exercise 4. Prove that $(\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a vector bundle.

Exercise 5. Show that the maps $E \oplus E' \to E$ and $E \oplus E' \to E'$ are bundle morphisms, and that there are two short exact sequences of vector bundles (i.e. diagrams of vector bundles which are exact on fibers):

$$0 \to E \to E \oplus E' \to E' \to 0$$
$$0 \to E' \to E \oplus E' \to E \to 0$$

Remark. In general, a short exact sequence of vector bundles would be written as

$$0 \to E \to F \to E' \to 0.$$

The preceding example is called a split short exact sequence as here we can write $F \cong E \oplus E'$ with a map which respects the inclusion $E \to F$ and projection $F \to E'$. In the appropriate topological and smooth categories, i.e. where we consider vector bundles with continuous or smooth transition maps and morphisms (defined over the correct domain of course), every short exact sequence is split due to the existence of Euclidean and Hermitian metrics, over $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ respectively. This is not the case in places like complex geometry, where splittings in the smooth category almost never respect the holomorphic structure.

Another way of yielding new vector bundles from old is that of the pullback bundle. Given $\pi : E \to B$ a vector bundle over B and $f : B' \to B$, as a topological space we can form the pullback square:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

The space $f^*E = \{(v, b) \in E \times B' : f(v) = \pi(b)\}$ so that the projection $f^*E \to B'$ is given by the restriction of $E \times B' \to B'$ and the fiber $f^*E_{b'} = E_{f(b)}$; this is the most natural operation to do, we just associate to each point in the domain of f the fiber of the point that f takes it to.

Exercise 6. Prove that f^*E is a vector bundle over B'.

To a pair of vector bundles E and E', we can define $\mathrm{Hom}(E, E')$ as the total space of a vector bundle with fiber $\mathrm{Hom}(E, E')_p := \mathrm{Hom}(E_p, E'_p)$.

Exercise 7. Prove that $\mathrm{Hom}(E, E')$ is a vector bundle, and the space of vector bundle homomorphisms $E \to E'$ is naturally isomorphic (as a vector space) to the space of sections of $\mathrm{Hom}(E, E')$.

3 Oriented Vector Bundles and the Euler Class

3.1 Euclidean and Oriented Vector bundles

Exercise 8. Prove that a real line bundle is trivial if and only if it is orientable.

3.3 Axiomatics of the Euler Class

Exercise 9. Prove that there is a one to one correspondence between characteristic classes, i.e. an assignment of $k(E) \in H^*(B)$ for every isomorphism class of $\mathbb{K}$-vector bundles $E \to B$ which is natural $k(f^*E) = f^*k(E) \in H^*(B')$ for every map $f : B' \to B$, and natural transformations between the $\mathrm{Vect}_{\mathbb{K}}$ functor and the cohomology functor $H^*(-)$.

4 Complex Vector Bundles and Chern Classes

4.1 Hermitian Vector Bundles

Exercise 10. Prove that for any B a paracompact space, and $E \to B$ a complex vector bundle, E admits a hermitian structure.

4.4 The Splitting Principle

The aim of this problem session is a loosely guided proof of Theorem 4.7. Much of the coverage is inspired by the approach of Bott & Tu. There are several different proofs, with varying levels of concreteness and constructiveness. Our proof will focus on the method of the splitting principle, a method which generalizes in a particularly nice way to the algebraogeometric setting. The basic idea is as follows:

i) Given a complex vector bundle $p : E \to B$ construct a space (in reality a fibration) $\pi : S \to B$ such that $\pi^*E \cong \bigoplus_{i \leq \mathrm{rk}(E)} L_i$ where each L_i is a line bundle over S.

ii) We then wish to prove that the induced map on cohomology $\pi^* : H^*(B, \mathbb{Z}) \to H^*(S, \mathbb{Z})$ is injective.

iii) Then we can produce the Chern class by forcing the product formula to hold, i.e. define $c(E)$ so that $\pi^* c(E) = \prod_i (1 + c_1(L_i))$.

iv) Then prove well definedness.

Before we delve into the splitting principle, we need to recall the structure of the cohomology of $\mathbb{P}(\mathbb{C}^k)$. Given $V \to \mathbb{C}^k$ a $k-1$ dimensional subspace, the image of $V \backslash \{0\}$ under the quotient map $\mathbb{C}^k \backslash \{0\} \to \mathbb{P}(\mathbb{C}^k)$, denoted by $\mathbb{P}(V)$ is homeomorphic to a copy of $\mathbb{P}(\mathbb{C}^{k-1})$. The complement $\mathbb{P}(V)^c$ is homeomorphic to $\mathbb{C}^{k-1}$. Thinking of $\mathbb{P}(C^2)$ as the one point compactification of $\mathbb{C}$, i.e. the CW complex obtained by attaching a 2-cell to $*$, we can use this process to inductively put a cell structure on $\mathbb{P}(\mathbb{C}^k)$, by attaching a $2k-2$ cell to $\mathbb{P}(\mathbb{C}^{k-1})$. As such the cellular homology cochain complex for $\mathbb{P}(\mathbb{C}^k)$ is

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \ldots \to 0 \to \mathbb{Z} \to 0$$

where the non-zero groups are in grading $2i$ for $2k-2 \geq 2i \geq 0$. Since the differentials are all 0, we can write the cohomology of $\mathbb{P}(\mathbb{C}^k)$ as $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \cong \mathbb{Z}[c_1]/c_1^{2k}$ where $c_1 = c_1(\tau)$ is placed as the generator in degree 2. Infact, this isomorphism respects the ring structure.

Proposition 4.1. *Let $p : E \to B$ be a complex vector bundle. There exists $\pi : S \to B$ such that $\pi^* E \cong \bigoplus_i L_i$ and π^* is injective on cohomology.*

Proof. Let $\pi : E \to B$ be a complex vector bundle. Define $\mathbb{P}(E)$ to be the fiber bundle with $\mathbb{P}(E)_p = \mathbb{P}(E_p)$. We can see that this is a fiber bundle over B by projectivizing the transition functions coming from a system of local trivializations of $E \to B$. Each of the fibers of $\mathbb{P}(E)$ carry the tautological line bundle $\tau \to \mathbb{P}(E_p)$ and the these glue together to give a line bundle $\tau(E) \to \mathbb{P}(E)$. This yields the following short exact sequence of vector bundles

$$0 \to \tau(E) \to \pi_1^* E \to \pi_1^* E/\tau(E) \to 0.$$

Introducing a hermitian metric allows us to split this short exact sequence (WARNING: in the algebraic category not all short exact sequences of vector bundles split, and in fact a weaker version of axiom a is used, that any short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ has $c(E) = c(E') \cup c(E''))$

Splitting this sequence allows us to write $\pi_1^* E \cong \tau(E) \oplus \pi^* E/\tau(E)$, now having split $\pi^* E$ into a sum of $\tau(E)$ and $\pi^* E/\tau(E)$ we can apply the process inductively to produce $S \to B$ for which $\pi^* E \cong \bigoplus L_i$. Denote our repeated projectivization $\pi : X_E \to B$

Exercise 11. Describe the fiber of this repeated process of projectivization as a quotient of some Lie group by one of its subgroups.

In order to show that the projection is injective on cohomology we may apply the following theorem:

Theorem 4.2. *(Leray-Hirsch) Let* $X \to B$ *be a fiber bundle with fiber* $i : F \to X$ *such that* $H^*(F, \mathbb{Z})$ *is a free abelian group and* $i^* : H^*(X, \mathbb{Z}) \to H^*(F, \mathbb{Z})$ *is surjective. Then there exists an isomorphism* $L : H^*(B, \mathbb{Z}) \otimes H^*(F, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ *such that the following diagram commutes*

$$\begin{array}{ccc} H^*(B,\mathbb{Z}) \otimes H^*(F,\mathbb{Z}) & \xrightarrow{L} & H^*(X,\mathbb{Z}) \\ \uparrow f & \nearrow \pi^* & \\ H^*(B,\mathbb{Z}) & & \end{array}$$

where $f(b) = b \otimes 1$.

So, if we can prove the hypotheses of Theorem 4.2 then the injectivity of f implies the injectivity of π^*. Proving these hypotheses can be rather tedious and uninstructive, so for the sake of brevity we will work in the case where a system of local trivializations has only two elements $U, V \subset B$.

Proposition 4.3. *The cohomology of* F, $H^*(F, \mathbb{Z})$ *is a free abelian group.*

Proof. Since each level is gotten by projectivizing the quotient of the previous bundle, it suffices to prove the following:

Lemma 4.4. *Let* $E \to B$ *be a complex vector bundle. Then the cohomology of* $\mathbb{P}(E)$ *is a freely generated module over* $H^*(B, \mathbb{Z})$, *generated by* $1, x, x^2, \ldots, x^{\mathrm{rk} E - 1}$ *where* $x \in H^2(\mathbb{P}(E), \mathbb{Z})$ *with* $\iota^* x = c_1(\tau)$.

Proof. Let $\iota : \mathbb{P}(\mathbb{C}^k) \to \mathbb{P}(E)$ denote the inclusion of a fiber. Naturality of $c_1(\tau)$ tells us that since $\iota^*\tau_E = \tau$ that $\iota^* c_1(\tau_E) = c_1(\tau)$. Since $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z})$ is a ring homomorphism, and $H^*(\mathbb{P}(\mathbb{C}^k), \mathbb{Z}) \cong \mathbb{Z}[c_1]/c_1^k$, ι^* is necessarily surjective and the Leray-Hirsch theorem applies. □

Definition 4.5. Since $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(B, \mathbb{Z})$, generated by $1, x, x^2, \ldots, x^{\text{rk}E-1}$ The equation

$$x^k = \sum_{i=1}^{k-1} a_i x^i$$

has a unique solution for some $a_i = \pi^* b_i$. We define the ith Chern class as $c_i(E) = -b_{k-i}$.

We can think of this idea as telling us that the cohomology of $\mathbb{P}(E)$ is additively $H^*(M, \mathbb{Z}) \otimes H^*(\mathbb{P}(\mathbb{C}^k))$, and the Chern classes measure the overall twisting of the ring structure on the cohomology. This gives a presentation of $H^*(\mathbb{P}(E), \mathbb{Z}) \cong H^*(B, \mathbb{Z})[x]/(\sum_{i=1}^{\text{rk}E} c_i(E) x^{\text{rk}E-i})$. □

Exercise 12. Using the previous ideas, write a presentation for the ring structure cohomology of $\mathbb{P}(E/\tau)$ as a quotient of $H^*(B, \mathbb{Z})[x_1, x_2]$.

Exercise 13. Set $B = U \cup V$ and $p^{-1}(U) \cong F \times U, p^{-1}(V) \cong F \times V$. Using the Mayer Vietoris sequence for $E = p^{-1}(U) \cup p^{-1}(V)$, and the Künneth formula, prove that for $x \in U \cap V$, $i^* : H^*(F, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is surjective.

□

Proceeding by induction, the preceding work tells us that as a ring $H^*(X, \mathbb{Z}) \cong H^*(B, \mathbb{Z})[x_1, \ldots, x_{\text{rk}E}]/I$ for some ideal $I \subset H^*(X, \mathbb{Z})[x_1, \ldots, x_{\text{rk}E}]$. We have a submersion $\Lambda_j : X \to \mathbb{P}(E)$ which sends the point $L_1 \oplus \ldots L_k$ to $[L_j]$. Since $\Lambda_j^* \tau = L_j$, $\Lambda_j^* x = c_1(L_j) = x_j$ and hence $c_1(L_j)$ satisfies

$$\sum_{i=1}^{k} x_j^i c_{\text{rk}E-i}(E) = 0$$

for every j. The theory of symmetric polynomials tells us that

$$c_i(E) = (-1)^i \sigma_i(x_1, \ldots, x_{\text{rk}E})$$

for σ_i the ith elementary symmetric polynomial

$$\sigma_i(x_1, \ldots, x_{\mathrm{rk}E}) = \sum_{I \subset [\mathrm{rk}E], |I|=i} x^I$$

so that $\sum c_i(E) = \sum \sigma_i(x_1, \ldots, x_{\mathrm{rk}E}) = \prod_{j=1}^{k}(1 + x_j) = \prod_{j=1}^{k}(1 + c_1(L_j))$ so that our definitions coincide.

Exercise 14. Prove the product formula $c(E \oplus E') = c(E)c(E')$ using the splitting principle.

Exercise 15. For L a complex line bunlde and E rank 2 complex vector bundle, find a formula for $c(L \otimes E')$.

5 Principal Bundles, Connections, and Curvature

5.1 Principal Bundles

Example 3. The associated bundle construction allows us to obtain all of the bundles related to a given principal G bundle by studying its representation theory. Given $\pi : E \to B$ a vector bundle and $FE \to B$ its bundle of frames, we can form the dual vector bundle $E^* = \mathrm{Hom}(E, \mathbb{K})$ by taking the bundle associated to the dual representation of $GL_n(\mathbb{K})$. More concretely, the action of $GL_n(\mathbb{K})$ on $\mathbb{K}^n$, $(g, v) \mapsto gv$ dualizes, to yield an action of $GL_n(\mathbb{K})$ on $(\mathbb{K}^n)^*$ by $(g, \alpha) \mapsto (v \mapsto \alpha(gv))$. This action is a right action, so we precompose by the inverse map to yield a left action. Then $E^* \cong (FE \times (\mathbb{K}^n)^*)/GL_n(\mathbb{K})$. For each $b \in B$ we have a natural evaluation map $FE_b \times \mathbb{K}^n \times (\mathbb{K}^n)^* \to FE_b \times \mathbb{K}$, given by $(p, v, \alpha) \mapsto (p, \alpha(v))$. This extends to a map $(FE\times\mathbb{K}^n)\times_\pi(FE\times K^n) \to FE \times \mathbb{K}$ which is $GL_n(\mathbb{K})$ equivariant (the action on $\mathbb{K}$ being trivial) since $g(p, v, \alpha) = (gp, g^{-1}v, \alpha \circ g) \mapsto (gp, \alpha(gg^{-1}v)) = (gp, \alpha(v))$. Equivariance means this descends to a map $E \times_\pi (FE \times_\rho (\mathbb{K}^n)^*) \to B \times \mathbb{K}$. This map is bilinear and therefore gives $\mathrm{ev} : FE \times_\rho \mathbb{K}^n \to \mathrm{Hom}(E, \mathbb{K})$

Remark. The associated bundle $P\times_\rho F$ should not be confused with a fibered product. The notation $P \times_\rho F$ is less than optimal, but it is rather compact.

Exercise 16. Prove that ev is an isomorphism.

5.2 Connections on Principal Bundles

Remark. We can extend the exterior derivative to any trivial vector bundle in a natural way: Given $\alpha \in \Omega^k(P, V)$, choosing isomorphism $\varphi : V \to \mathbb{R}^n$ means we can realize α as an element α' of $\Omega^k(P, \mathbb{R}^n)$ or as an n-tuple of forms $\alpha'_i := ((v_1, \dots, v_k) \mapsto \Phi(\alpha(v_1, \dots, v_k))_i)$ then $d\alpha := \Phi^{-1}(d\alpha')$.

Exercise 17. Show that the extension of d to $\Omega^k(P, V)$ doesn't depend on our choice of Φ. Show that as a module over $\Omega^*(P)$, d defines a (super)-derivation of degree 1 on $\Omega^*(P, V)$.

We can view a connection as a sort of universal covariant derivative on vector bundles associated to $P \to B$ in the following way. Given a linear representation $\rho : G \to GL(V)$, the associated bundle construction gives rise to an equivalence between sections of $E = P \times_\rho V \to B$ and equivariant functions $P \to V$, and E-valued forms with equivariant V-valued forms which vanish on the vertical distribution. Then, given an E-valued k-form η, with corresponding form on P denoted $\tilde{\eta}$, we can take its exterior derivative $d\tilde{\eta}$. Using our connection $D\tilde{\eta} = d\tilde{\eta} \circ h$ is an equivariant V-valued $k+1$ form on P which vanishes on the vertical distribution and hence descends to a unique E-valued $k+1$ form $d_\omega \eta$.

Exercise 18. Prove that $d_\omega : \Omega^*(B, E) \to \Omega^*(B, E)$ is a degree $+1$ derivation (viewing $\Omega^*(B, E)$ as a module over $\Omega^*(B)$).

5.3 Chern-Weil Theory

Proposition 5.1. *For* $\tau \to \mathbb{P}(\mathbb{C}^2)$*,* $w_\tau(f) = -[\mathbb{P}(\mathbb{C}^2)] \in H^2(\mathbb{P}(\mathbb{C}^2), \mathbb{R})$

Proof. By equipping $\mathbb{C}^2$ with the standard hermitian form, and seeing that $F\tau = \mathbb{C}^2 \backslash \{0\}$, we have $S\tau = S^3$ equipped with the inverse of the standard $U(1)$ action, $(\exp(i\theta), z) \mapsto \exp(-i\theta)z$. With respect to the canonical trivialization of $T\mathbb{C}^k|_{S^3}$, the $U(1)$ action is generated by the vector field $z \mapsto -iz$ (since $\exp(-it)z$ differentiates to $-i\exp(it)z$). Because the standard inner product on $\mathbb{C}^2 \cong \mathbb{R}^4$ is given by $\mathrm{Re}(h)$ we have that $TS^3 = \ker(\langle -, z\rangle)|_{S^3}$. An Ehresmann connection on $S^3 \to S^2$, can be gotten by taking the orthogonal compliment to the vertical bundle with respect to the standard inner product, hence $H = \ker(\mathrm{Re}(h(-, iz))|_{TS^3}$. If $\sigma \to S^3$ is the subbundle of $\mathbb{C}^2 \times S^3$ written as $\{(\lambda z_1, \lambda z_2, z_1, z_2) | (z_1, z_2) \in S^3, \lambda \in \mathbb{C}\}$, then $H = \sigma^\perp$. We can then write our connection form as

$$\omega = \mathrm{Re}(h(-, iz)) = -i(y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2).$$

Over the chart U_1 of $\mathbb{P}(\mathbb{C}^2)$, we have a section of $S^3 \to \mathbb{P}(\mathbb{C}^2)$ given by $s : z \mapsto \left(\frac{z}{\sqrt{|z|^2+1}}, \frac{1}{\sqrt{|z|^2+1}}\right)$. Under this section we have $s^*\omega = -i\frac{ydx - xdy}{x^2+y^2+1}$ so that $\Omega = ds^*\omega = \frac{-2ir^2 dr \wedge d\theta}{(r^2+1)^2}$ in polar coordinates. Since $U_1^c = \{[1 : 0]\}$, integration over U_1 will yield the value of the whole integral, and

$$\langle c_1(\tau), [\mathbb{P}(\mathbb{C}^2)]\rangle = \int_{\mathbb{P}(\mathbb{C}^2)} \frac{1}{2\pi i}\Omega = \frac{1}{2\pi i}\int_{\mathbb{C}} -\frac{r^2}{(r^2+1)^2} dr \wedge d\theta = -1$$

as desired. □

Exercise 19. Fill in the details of the preceding calculation.

For Chern classes, we will make use of the existence of a hermitian metric, which makes every complex vector bundle associated to a principal $U(n)$ bundle. The theory of polynomials tells us that invariant symmetric k-multilinear functions $\mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$ are the same as adjoint invariant degree k polynomials on $\mathfrak{g}$. For $\lambda \in \mathbb{R}$, consider the polynomial $\det(\lambda I + iX) = \lambda^n - f_1(X)\lambda^{n-1} + f_2(X)\lambda^{n-2} + \ldots + (-1)^n f_n(X)$ for $X \in \mathfrak{u}(n)$ (does this polynomial look familiar?)

Proposition 5.2. *Each* $f_i \in I(U(n))$ *and the collection* $\{f_i\}_{i \leq n}$ *generates* $I(U(n))$.

Exercise 20. Prove that $f_i \in I(U(n))$ for $0 \leq i \leq n$.

For a proof of the second fact, see Kobayashi-Nomizu Vol II.

Part of the power of Chern-Weil theory, is that it allows us to prove identities about characteristic classes by doing matrix algebra.

Proposition 5.3. *The Chern-Weil theory Chern class satisfies the product formula, i.e.* $c^{ChWe}(E \oplus E') = c^{ChWe}(E)c^{ChWe}(E')$

Exercise 21. Prove Proposition 5.3