Preliminary Exam in Algebra-Fall Semester

August 2011

Do three of the following five problems.

Question 1. *(a) Let G be a group of order 12. Prove that either the Sylow 2-subgroup or Sylow 3-subgroup of G is normal. (b) Show that a group G of order 12 with no subgroup of order 6 is isomorphic to A_4. (If needed you may assume without proof that $Aut(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong S_3$)*

Question 2. *Let R be a commutative ring with identity. The Jacobson Radical of R, denoted $\mathcal{J}$ is the intersection of all maximal ideals in R. (a) Compute the Jacobson Radical of the ring $\mathbf{Z}/30\mathbf{Z}$. (b) Show that $x \in \mathcal{J}$ if and only if $1 - xy$ is a unit in R for all $y \in R$.*

Question 3. *Show that for G a finite abelian group, any irreducible n-dimensional complex representation is one-dimensional.*

Question 4. *Let R be a commutative ring with 1. If $a \in R$, write $Ann(a) = \{r \in R : ar = 0\}$, an ideal of R. Let $S \subset R$ be the set of $a \in R$ with $Ann(a)$ a prime ideal. (a) If R is Noetherian, show that S is nonempty. (b) If $a \in S$ and $r \in R$, show that either $ar = 0$ or $ar \in S$. (c) If $a, b \in S$ and $Ann(a) \neq Ann(b)$, show that $ab = 0$.*

Question 5. *Let G be a group and $K \subset H$ subgroups with K normal in H. (a) Prove that H normalizes the centralizer $Z_G(K)$. (b) If H is normal in G and $Z_H(K) = 1$, prove that H centralizes $Z_G(K)$.*