Preliminary Exam in Algebra-Spring Semester

August 2011

Do three of the following four problems.

Question 1. *Prove that the* $\sqrt[5]{7}$ *does not lie in* $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ *where* $\alpha_1^3, \ldots, \alpha_n^3 \in \mathbb{Q}$.

Question 2. *Let* p *be a rational prime and* $a \in \mathbb{F}_p^*$.

a) Prove that $x^p - x + a$ *is irreducible and separable over* $\mathbb{F}_p$.

b) Determine the number of elements of the splitting field of $x^p - x + a$ *over* $\mathbb{F}_p$.

Question 3. *Let* F *be the splitting field over* $\mathbb{Q}$ *of* $x^4 + x^2 + 1$.

a) Determine the degree of $F/\mathbb{Q}$.

b) Detemine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ *as a subgroup of the permutation group of the roots of* $x^4 + x^2 + 1$.

c) Determine a primitive generator for all the intermediate fields of $F/\mathbb{Q}$

Question 4. *Let* p *be a prime.*

a) Prove that the Galois group of $x^p - 2$ *is isomorphic to the group of matrices* $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ *where* $a, b \in \mathbb{F}_p$ *and* $a \neq 0$.

b) Prove that $\mathbb{Q}(\sqrt[p]{2})$ *is not a subfield of any cyclotomic extension of* $\mathbb{Q}$ *for any odd prime* p.