PRELIMINARY EXAMINATION IN ANALYSIS
Part I, Real Analysis
August 15, 2011

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1. For $\frac{1}{p} + \frac{1}{q} = 1$, let $S = \big\{ f \in \mathrm{L}^p(\mathbb{R}) : \text{support}(f) \subset [-1,1],\ \text{and } \|f\|_{\mathrm{L}^p} \le 1 \big\}$, and let g be a fixed but arbitrary function in $\mathrm{L}^q(\mathbb{R})$, with support$(g) \subset [-1,1]$. Show that the image of S under the map $f \mapsto f * g$ is a compact set in $\mathrm{C}^0\big([-2,2]\big)$.

2. Let $f_1, f_2, f_3, \ldots$ be nonnegative Lebesgue-integrable functions on $\mathbb{R}^n$, such that

$$\sum_{k=1}^{\infty} \int (f_k - f_{k-1})^+ < \infty \,, \qquad \lim_{k\to\infty} \int f_k \ = 0 \,.$$

Show that $\limsup_{k\to\infty} f_k \equiv 0$ almost everywhere.

3. Let $1 < p < \infty$ and $f(x) = |x|^{-n/p}$ for $x \in \mathbb{R}^n$. Prove that f is <u>not</u> the limit of a sequence $f_k \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ in the sense of convergence in $\mathrm{L}^p_{\text{weak}}(\mathbb{R}^n)$.
$\Big($ That is, $\limsup_{k\to\infty} \big(\sup_{\lambda>0} \lambda^p \big| \{ x \in \mathbb{R}^n : |f(x) - f_k(x)| > \lambda \} \big| \big) > 0$ for any such sequence.$\Big)$

4. Let μ be a Borel measure on $[0,1]$. Assume that
a) μ and Lebesgue measure are mutually singular.
b) $\mu\big([0,t]\big)$ depends continuously on t.
c) For any function $f : [0,1] \to \mathbb{R}$, if $f \in \mathrm{L}^1(\text{Lebesgue})$ then $f \in \mathrm{L}^1(\mu)$.
(Note that f has a finite value at every point.)
Show that $\mu \equiv 0$.