Name and UID:

Applied Math Prelim

August 17, 2011

Part I

1. Let X, Y, and Z be NLS's and $T : X \times Y \to Z$ a bilinear map.
(a) Prove that the following are equivalent.

(i) T is continuous;

(ii) T is continuous at $(0, 0)$;

(iii) T is bounded, meaning that there is some $M \geq 0$ such that

$$\|T(x, y)\|_Z \leq M \|x\|_X \|y\|_Y \quad \forall x \in X,\ y \in Y.$$

(b) Show that the minimal M above gives a norm on the set of continuous bilinear maps. That is, $\|\cdot\|$ is a norm, where

$$\|T\| = \sup_{x \in X, y \in Y} \frac{\|T(x, y)\|_Z}{\|x\|_X \|y\|_Y}.$$

2. Let H be a nontrivial Hilbert space. Let $P : H \to M$ be a linear projection operator, and let $Q : H \to N$ be an *orthogonal* projection operator. Assume that M and N are neither $\{0\}$ nor H.

(a) Prove that $\|P\| \geq 1$.
(b) Prove that $\|P\| = 1$ if and only if P is an orthogonal projection.
(c) Suppose now that P is an orthogonal projection, and also that $PQ = QP$. Show that PQ is an orthogonal projection onto $M \cap N$.

3. Let $K : L^2(0, 1) \to L^2(0, 1)$ be the integral operator defined as

$$Ku(x) = \int_0^1 e^{x-y} u(y) dy.$$

(a) Find the range of K. Is the range of K closed? Is K a compact operator?

(b) Compute the adjoint operator K^*, and find its kernel.

(c) Verify explicitly that $Ku = f$ is solvable if and only if $f \perp \text{Ker } K^*$.

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Part II

Choose any 3 of the 4 following problems.

4. For $\varphi \in C^0(\mathbb{R}^d)$, let the restriction map $R : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^{d-k})$ be defined by $R\varphi(x') = \varphi(x', 0)$, $\forall x' \in \mathbb{R}^{d-k}$ and $0 \in \mathbb{R}^k$, for $0 < k < d$, with k an integer number.

Show that the restriction map R extends to a bounded linear map from $H^s(\mathbb{R}^d)$ onto $H^{s-k/2}(\mathbb{R}^{d-k})$, provided that $s > k/2$.

Hint: Show this result first for the restriction of functions in $\mathcal{S}(\mathbb{R}^d)$ using the Sobolev norms involving the Fourier representation.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$ and $\alpha > 0$. Consider the Robin boundary value problem in Ω,

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) For this problem, formulate a variational principle $B(u, v) = (f, v), \forall v \in H^1(\Omega)$.

(b) Show that this problem has a unique weak solution.

6. Set up and apply the contraction mapping principle to show that the boundary value problem ($\varepsilon > 0$):

$$\begin{cases} -u_{xx} + u - \varepsilon u^2 = f(x), \ x \in (0, +\infty), \\ u(0) = 1, \qquad \lim_{x \to +\infty} u(x) = 0, \end{cases}$$

where $f(x)$ is a smooth compactly supported function on $(0, +\infty)$, has a unique smooth solution if ε is small enough.

7. Show that for $y \in \mathbb{R}^2$ fixed, $\frac{1}{2\pi} \ln |x - y|$ is locally integrable in $\mathbb{R}^2$, i.e. it is a function in $L_{1,loc}(\mathbb{R}^2)$; and that it is a fundamental solution of $\Delta u = \delta_y$, where $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ is the Laplace operator.