Numerical Analysis Exam: Part A, August 2011

1. Let $f(x)$ be a smooth function.

   (a) Write down the Newton's method for root finding.

   (b) Assume that $f(\xi) = f'(\xi) = f''(\xi) = 0$. If (x_k) is a sequence obtained by Newton's method, prove that

   $$\xi - x_{k+1} = (\xi - x_k)\left(1 - \frac{1}{3}\frac{f'''(\alpha_k)}{f'''(\beta_k)}\right)$$

   where α_k and β_k lie between ξ and x_k.

   (c) Suppose $0 < m < |f'''(x)| < M$ in a neighborhood $[\xi - \eta, \xi + \eta]$ for $\eta > 0$. Find a condition on m and M under which the Newton method converges for any initial condition from $[\xi - \eta, \xi + \eta]$. State the convergence rate.

2. Suppose that A is a matrix of size $m \times n$ with $m > n$.

   (a) Show how to solve the least square problem

   $$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|^2$$

   using the QR decomposition and the singular value decomposition (SVD), respectively. Notice that the rank of A may be smaller than n.

   (b) Show how to solve the regularized problem

   $$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|^2 + \frac{1}{2}\alpha\|x\|^2$$

   with $\alpha > 0$ using the singular value decomposition (SVD).

3. Suppose that $P_1(x), \ldots, P_n(x)$ are n linear independent functions on a domain Ω.

   (a) Suppose that (x_j, u_j) for $j = 1, 2, \ldots, n$ are given for $\{x_j\}$ different. What is the linear system for finding $u(x) \in \text{span}\{P_1(x), \ldots, P_n(x)\}$ that interpolates (x_j, u_j), i.e., $u(x_j) = u_j$.

   (b) Suppose that $g(x)$ is a function defined on Ω. What is the linear system for finding $u(x) \in \text{span}\{P_1(x), \ldots, P_n(x)\}$ that solves

   $$\min_u \frac{1}{2}\int_\Omega |g(x) - u(x)|^2 dx.$$

   (c) Assume that Ω is the periodic interval $[0, 1]$, $x_j = (j-1)/n$ for $j = 1, \ldots, n$, and $\{P_i(x), i = 1, \ldots, n\} = \{e^{2\pi\sqrt{-1}kx}, k = -m, \ldots, m\}$ where $n = 2m + 1$. Write down explicitly the linear system of the previous two questions.

   (d) How to solve these two problems efficiently in $O(n \log n)$ steps? [Hint: Use trapezoidal rule (for quadrature) and FFT.]