PRELIMINARY EXAMINATION: NUMERICAL ANALYSIS II

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Work all 3 of the following 3 problems.

1. Consider the system of two ordinary differential equations

$$u'(t) = f(v) \quad \text{and} \quad v'(t) = g(u),$$

where $u(0) = u_0$ and $v(0) = v_0$, and, for $h > 0$, the numerical scheme

$$U^{n+1} = U^n + h\, f\Big(V^n + \frac{h}{2} g(U^n)\Big) \quad \text{and} \quad V^{n+1} = V^n + h\, g\Big(\frac{1}{2}(U^n + U^{n+1})\Big).$$

(**a**) Show that the local truncation error for both u and v is $\mathcal{O}(h^2)$.

(**b**) For the linear system where $f(v) = \lambda v$ and $g(u) = -\mu u$, both λ and μ being positive, show that when $h\lambda < 1$ and $h\mu < 1$, the scheme is stable. [Hint: The eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$ are $a \pm \sqrt{bc}$.]

2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal boundary. Consider the elliptic partial differential equation for $u(x)$ given by

$$\begin{aligned} -a\Delta u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $a(x)$ and $c(x)$ satisfy $0 < a_* \le a(x) \le a^* < \infty$, $0 \le c(x) \le a^* < \infty$, and also $|\nabla a(x)| \le b^* < \infty$. Assume that $f \in L^2(\Omega)$.

(**a**) Find a variational form suitable for approximation by finite elements.

(**b**) Give a reasonable condition on b^* that insures that your bilinear form is coercive.

(**c**) Derive a bound on the error between u and a finite element approximation u_h.

3. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal boundary. Consider the parabolic partial differential equation

$$\begin{aligned} u_t - \Delta u &= f(x,t) \quad && \text{for } x \in \Omega,\ t > 0, \\ u(x,t) &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega,\ t = 0. \end{aligned}$$

It has the variational form

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

(**a**) Write down the discrete scheme that uses a suitable finite element method in space and backward Euler in time.

(**b**) Show that your scheme is stable by bounding

$$\max_n \|u^n\|^2 + \sum_n \|\nabla u^n\|^2 \,\Delta t$$

in terms of $\|u_0\|$ and $\max_t \|f(\cdot, t)\|$.