THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability Part I

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Problem 1 (36pts)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables bounded in $\mathcal{L}^1$. We say that $\{X_n\}_{n\in\mathbb{N}}$ converges to a random variable $X \in \mathcal{L}^1$ **in the biting sense** if, for each $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{F}$ such that $\mathbb{P}[A_\varepsilon] > 1-\varepsilon$ and $X_n \mathbf{1}_{A_\varepsilon} \to X \mathbf{1}_{A_\varepsilon}$ in $\mathcal{L}^1$.

(1) Show that the biting limit X of the sequence $\{X_n\}_{n\in\mathbb{N}}$ is a.s.-unique (provided it exists).
(2) Show that convergence in $\mathcal{L}^1$ implies the biting convergence, but that the biting convergence does not necessarily imply convergence in $\mathcal{L}^1$.
(3) Show that he two concepts coincide for uniformly integrable sequences $\{X_n\}_{n\in\mathbb{N}}$.

Problem 2 (32pts)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $\mathbb{P}[X_n \in \mathbb{N}_0] = 1$, for all $n \in \mathbb{N}$; here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Show that for a random variable X with $\mathbb{P}[X \in \mathbb{N}_0] = 1$, the following are equivalent

(1) $X_n \to X$ in distribution,
(2) $\mathbb{P}[X_n = k] \to \mathbb{P}[X = k]$, for all $k \in \mathbb{N}_0$.
(3) $g_{X_n}(t) \to g_X(t)$ for all $t \in [0, 1)$, where, for any Y, we define $g_Y(t) = \sum_{k\in\mathbb{N}_0} \mathbb{P}[Y = k]\, t^k$.

(*Hint:* For (3) $\Rightarrow$ (1), use the ideas from the proof of the Continuity Theorem for weak convergence. To deal with tightness, show that $\mathbb{P}[Y \geq N] \leq \frac{2}{\varepsilon}\int_0^\varepsilon (1 - g_Y(t))\, dt$, for any random variable Y with $\mathbb{P}[Y \in \mathbb{N}_0] = 1$ and any $N \in \mathbb{N}$, $\varepsilon > 0$ with $\varepsilon^N \leq \frac{1}{2}$.)

Problem 3 (32pts)**.** Let $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ be an iid sequence with $\mathbb{P}[\varepsilon_n = 1] = 1 - \mathbb{P}[\varepsilon_n = -1] = p \in (\frac{1}{2}, 1)$. We interpret $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ as outcomes of a series of gambles. A gambler starts with $Z_0 > 0$ dollars, and in each play wagers a certain portion of her wealth. More precisely, the wealth of the gambler at time $n \in \mathbb{N}$ is given by

$$Z_n = Z_0 + \sum_{k=1}^{n} C_k \varepsilon_k,$$

where $\{C_n\}_{n\in\mathbb{N}_0}$ is a predictable process such that $C_k \in [0, Z_{k-1})$, for $k \in \mathbb{N}$. The goal of the gambler is to maximize the "return" on her wealth, i.e., to choose a strategy $\{C_n\}_{n\in\mathbb{N}_0}$ such that the expectation $\frac{1}{T}\mathbb{E}[\log(Z_T/Z_0)]$, where $T \in \mathbb{N}$ is some fixed time horizon, is the maximal possible.

(1) Define $\alpha = H(\frac{1}{2}) - H(p)$, where $H(p) = -p\log p - (1-p)\log(1-p)$ is the entropy function, and show that the process $\{W_n\}_{n\in\mathbb{N}_0}$ given by
$$W_n = \log(Z_n) - \alpha n, \text{ for } n \in \mathbb{N}_0$$
is a supermartingale. Conclude that $\mathbb{E}[\log(Z_T)] \leq \log(Z_0) + \alpha T$, for any choice of $\{C_n\}_{n\in\mathbb{N}_0}$.
(2) Show that the upper bound above is attained for some strategy $\{C_n\}_{n\in\mathbb{N}_0}$.