THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part II

August, 2011

Problem 2.1. (35 points, Inverse of the 3-d Bessel Process). Consider a 3-dimensional Brownian Motion W starting at $W_0 = (1, 0, 0)$ and denote by

$$R_t := \|W_t\|,$$

and

$$T := \inf\{t |\ R_t = 0\}.$$

(1) Show that the process
$$L_t = \frac{1}{R_t}, \quad 0 \le t < T$$
is a local martingale
(2) Show that $\mathbb{P}[T = \infty] = 1$
(3) Sketch a brief argument (no complete computations needed) showing that L is actually a strict local martingale, which means that it is NOT a martingale (for $0 \le t < \infty$).

Problem 2.2. (40 points) Consider a standard one -dimensional Brownian Motion W and denote by T_b the first hitting time of level b. We know that

$$\mathbb{P}[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt, t > 0,$$

$$\mathbb{E}[e^{-\lambda T_b}] = e^{-|b|\sqrt{2\lambda}}, \quad \lambda > 0.$$

Let $\mu \in \mathbb{R}$ and consider the measure $\mathbb{P}_\mu$ under which the process

$$W_t^\mu = W_t - \mu t$$

is a Brownian Motion (so that the original W is a Brownian Motion with drift μ under $\mathbb{P}_\mu$).

(1) compute $\mathbb{P}_\mu[T_b \in dt]$, $t > 0$
(2) compute $\mathbb{P}_\mu[T_b < \infty]$
(3) if $\mu > 0$ and $W_* = \inf_{t \ge 0} W_t$, show that $-W_*$ has an exponential distribution with parameter 2μ under $\mathbb{P}_\mu$.

Problem 2.3. (25 points) Consider a standard one dimensional Brownian Motion W and the random variable

$$Y := \int_0^1 \sin t \, dW_t.$$

Show that Y is Gaussian. Compute its mean and variance.