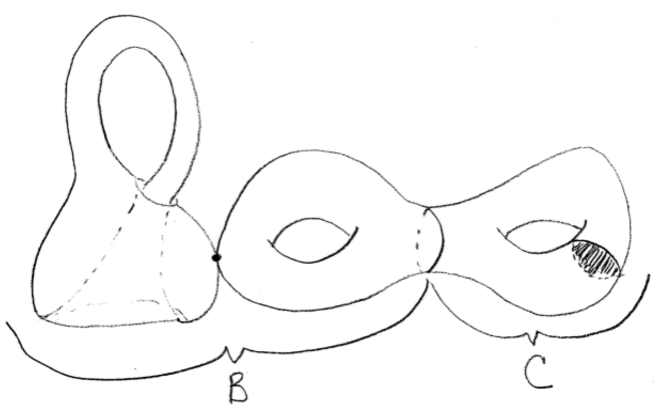
Preliminary Examination in Topology: August 2011

Algebraic Topology

1. All parts of this question refer to the pictured 2-complex $A = B \cup C$. The complex A consists of a Klein bottle wedge a double torus with a disk attached that spans a meridional curve. The complex A is the union of two subcomplexes B and C as pictured. These first sub-questions are designed to find out whether you know the fundamental groups of various parts of A.
   (a) What are the fundamental groups of the following:
      (i) a Klein bottle;
      (ii) a double torus, that is, the connected sum of two tori;
      (iii) a double torus with a disk attached that spans a meridional curve;
      (iv) the subcomplex B;
      (v) the subcomplex C.
   (b) Just by eye, write down a presentation of the fundamental group of A. What did you look at when reaching your conclusion?
   (c) Use the decomposition of A as the union $B \cup C$ (or, more exactly, as the union of small neighborhoods of them) and apply Van Kampen's Theorem to confirm your answer to part (b). You may use your responses to (a)(iv) and (a)(v) without further justification.



2. All parts of this question again refer to the same pictured 2-complex $A = B \cup C$. Again, the complex A consists of a Klein bottle wedge a double torus with a disk attached that spans a meridional curve. The complex A is the union of two subcomplexes B and C as pictured. These first sub-questions are designed to find out whether you know the homology groups of various parts of A.
   (a) What are the nth homology groups ($n = 0, 1, 2, 3$) of the following:
      (i) a Klein bottle;
      (ii) a double torus, that is, the connected sum of two tori;
      (iii) a double torus with a disk attached that spans a meridional curve;
      (iv) the subcomplex B;

(v) the subcomplex C.

(b) Just by eye, write down the nth homology groups of A ($n = 0, 1, 2, 3$). What did you look at when reaching your conclusion?

(c) Use the decomposition of A as the union $B \cup C$ and apply the Mayer-Vietoris Theorem to confirm your answer to part (b). You may use your responses to (a)(iv) and (a)(v) without further justification.

3. Let F be a free group on n generators, and let H be an index k subgroup of F.
   (a) The *rank* of a finitely-generated group is the minimal number of generators in a presentation of that group. Show that the rank of F is n.
   (b) Show that H is a free group.
   (c) Show that $Rank(H) - 1 = k(Rank(F) - 1)$.
   (d) Exhibit explicitly an index 3 subgroup of the free group on two generators a, b.

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Differential Topology

All manifolds are smooth and without boundary unless a boundary is explicitly allowed. All functions are smooth.

1. Identify $\mathbb{R}^2$, with coordinates x, y, with $\mathbb{C}$, with coordinate $z = x + iy$. Likewise, identify a copy of $\mathbb{R}^2$ with coordinates u, v with $\mathbb{C}$ with coordinate $w = u + iv$. Let $f : \{\mathbb{R}^2 - (1,0) - (-1,0)\} \to \mathbb{R}^2$ be the function $f(z) = \frac{1}{z-1} - \frac{1}{\bar{z}+1}$.
   (a) Show that f extends to a smooth map $\tilde{f} \colon S^2 \to S^2$, where S^2 (or $\mathbb{C}P^1$) is the one-point compactification of $\mathbb{R}^2$ (or $\mathbb{C}$).
   (b) Compute the degree of $\tilde{f}$.

2. (a) Suppose ω is a 1-form on $\mathbb{RP}^2$ which is closed: $d\omega = 0$. Let $f \colon S^1 \to \mathbb{RP}^2$ be a map. Prove that
   $$\int_{S^1} f^*\omega = 0.$$
   (b) Fix orientations on $S^3, \mathbb{RP}^3$ such that the covering map $p \colon S^3 \to \mathbb{RP}^3$ is orientation-preserving. Prove that there exists $\eta \in \Omega^3(\mathbb{RP}^3)$ such that
   $$\int_{S^3} p^*\eta = 1.$$

3. For each of the following statements sketch a proof or give a counterexample.
   (a) Let X be an oriented compact manifold and $f \colon X \to S^2$ a map which is transverse to a great circle $C \subset S^2$. Let $Z \subset X$ be any compact 1-dimensional submanifold. Then the oriented intersection number of Z with $f^{-1}(C)$ vanishes.
   (b) Let X be an oriented compact manifold and $f \colon X \to X$ a map. Suppose W is a compact oriented manifold with boundary $\partial W = X$ and $F \colon W \to W$ a map whose restriction to the boundary is f. Then the global Lefschetz number of f vanishes.