PRELIM EXAM IN ALGEBRA, SPRING 2012

Part I

Do three of the following five problems.

1. Let p be a prime.

1a. Classify elements of the symmetric group S_p of order p.

1b. Let G be a finite group and H be a subgroup of index p. Assume p is the smallest prime number that divides the order of G. Show that H is normal. [Hint: if H is not normal, how many conjugates will it have?]

2a. Show that any matrix in $\mathrm{GL}_n(\mathbb{C})$ of finite order is diagonalizable.

2b. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite abelian subgroup of the invertible n by n complex matrices. Show that G is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{C})$ whose elements are all diagonal matrices.

3. Show that, up to isomorphism, there are exactly four groups of order 170.

4a. Let A be an n by n matrix with entries in a field F, and suppose that the minimal polynomial of A is equal to the characteristic polynomial of A. Show that for any n by n matrix B with entries in F such that $BA = AB$, there exists a polynomial $P \in F[t]$ such that $B = P(A)$.

4b. Show that if the characteristic polynomial of A is *not* equal to the minimal polynomial of A, then there exists an n by n matrix B with entries in F such that $BA = AB$ but B is not equal to $P(A)$ for any polynomial P.

5. Let G be a p-group, and let H be a normal subgroup of G. Show that $H \cap Z(G)$ contains an element other than the identity.

Algebra Part II

1. Let F be a finite field of odd characteristic. Prove that the product of the nonzero elements of F is equal to -1.

2. Let E be the splitting field of the polynomial $x^5 - 3$ over $\mathbb{Q}$.

2a. Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ as a group of permutations of the roots of $x^5 - 3$.

2b. Prove that E is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

3. Consider $f(x) = x^4 + 7x + 7$.

3a. Determine the degree of the splitting field of $f(x)$ over $\mathbb{F}_3$.

3b. Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\mathrm{Gal}(F/\mathbb{Q}) \cong S_4$.

4. Let q be a prime power. Determine, in terms of q, the number of irreducible monic polynomials in $\mathbb{F}_q[t]$ of degree 6.

5. Let $f \in \mathbb{Q}[t]$ be an irreducible polynomial of degree 5, and let K be the splitting field of f over $\mathbb{Q}$.

5a. Suppose f has exactly three real roots. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to S_5.

5b. Now suppose f has exactly one real root. What can you say about $\mathrm{Gal}(K/\mathbb{Q})$?