PRELIMINARY EXAMINATION IN ANALYSIS
PART I - REAL ANALYSIS
JANUARY 6, 2012

(1) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0,1)$ there is an open interval I such that

$$|\,E \cap I\,| \geq \alpha |\,I\,|.$$

(2) Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $A = \{x^2 \mid x \in Z\}$ also has measure zero.

(3) Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that given $\varepsilon > 0$, there exists E, with $|E| < \varepsilon$, so that $f_k \to f$ uniformly on $I \setminus E$, for any finite interval I.

(4) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f \in L^1(\Omega)$. Prove that

$$\lim_{p\to 0} \left[\int_\Omega |f|^p \, d\mu \right]^{1/p} = \exp \left[\int_\Omega \log |f| \, d\mu \right],$$

where $\exp[-\infty] = 0$. To simplify the problem, you may assume $\log|f| \in L^1(\Omega)$.

(5) Let h be a bounded, measurable function, such that, for any interval I

$$\left| \int_I h \right| \leq |\,I\,|^{\frac{1}{2}}.$$

Let $h_\varepsilon(x) = h(\frac{x}{\varepsilon})$. Show that for any A with $|A| < \infty$

$$\int_A h_\varepsilon(x)\, dx \to 0, \text{ as } \varepsilon \to 0.$$