PRELIMINARY EXAMINATION IN ANALYSIS
PART II - COMPLEX ANALYSIS
JANUARY 6, 2012

(1) (a) What can you say about an entire function F that does not take any real value in$[-1,1]$? Justify your answer.

(b) What can you say about an entire function f whose real part is always less than its imaginary part? Justify your answer.

(2) Let Ω be a bounded, open, connected subset of $\mathbb{C}$ and let $\phi : \Omega \to \Omega$ be an analytic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\phi(z_0) = z_0 \quad \text{and} \quad \phi'(z_0) = 1,$$

then ϕ is linear.

(3) Let $\varepsilon > 0$ and $\omega \in \mathbb{C}$. Show that the function $z \mapsto \sin(z) + \frac{1}{\omega+z}$ has infinitely many zeros in the strip $|\operatorname{Im} z| < \varepsilon$.

(4) If f is analytic in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$, prove that there is a sequence $\{z_n\}$ in D, approaching the boundary of D, such that the sequence $\{f(z_n)\}$ is bounded.

(5) Assume $g : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and extends analytically to a complex open neighborhood of $\mathbb{R}$. Show that there exists a bounded analytic function f in the upper half plane, and a bounded analytic function h in the lower half plane, such that

$$g(x) = \lim_{y \downarrow 0} \big[f(x+iy) - h(x-iy) \big] \,, \qquad x \in \mathbb{R}\,.$$