PRELIMINARY EXAMINATION: APPLIED MATHEMATICS I

January 11, 2012, 1:00-2:30

Work all 3 of the following 3 problems.

1. State and provide proofs for Hölder and Minkowski inequalities in $L^p(\Omega)$ spaces, for $1 \le p \le \infty$.

2. Let H be a Hilbert space and M a closed linear subspace. Let $A : H \to H$ be a bounded linear operator with a bounded inverse.

(**a**) For a given fixed $x \in H$, show that there is a unique $y \in M$ such that

$$\inf_{z \in M} \|A(z-x)\| = \|A(y-x)\|.$$

This defines an operator $P : H \to M$ by $Px = y$. [Hint: Use the parallelogram law.]

(**b**) Show that

$$\langle A^* A(Px - x), y \rangle = 0 \quad \forall y \in M.$$

[Hint: Consider for $\lambda \in \mathbb{F}$ and $y \in M$ that $\|A(Px-x)\|^2 \le \|A(Px - x - \lambda y)\|^2$, and find a good choice for λ.]

(**c**) Show that P is bounded, with bounding constant $\|A\| \, \|A^{-1}\|$.

3. Let H be a separable Hilbert space with maximal orthonormal basis $\{u_k\}_{k=1}^{\infty}$, let $H_n = \text{span}\{u_1, \ldots, u_n\}$, and let P_n denote the orthogonal projection of H onto H_n. Suppose that $A : H \to H$ is bounded and linear and $f \in H$. If

$$P_n A x_n = P_n f$$

has a solution $x_n \in H_n$ such that

$$\|x_n\| \le \alpha \|P_n f\|,$$

where $\alpha > 0$ is independent of n, show that there is at least one solution to $Ax = f$.

PRELIMINARY EXAMINATION: APPLIED MATHEMATICS II

January 11, 2012, 2:40-4:10

Work all 3 of the following 3 problems.

1. Consider the bilinear form

$$B(u,v) = (\nabla u, \nabla v)_{L_2(\Omega)} + (bu, \nabla v)_{L_2(\Omega)} + (cu, v)_{L_2(\Omega)}$$

(**a**) Let c be positive constant. Derive a relation between b and c to insure that the bilinear form B is coercive on $H^1(\Omega)$.

(**b**) Suppose $b = 0$. If $c < 0$, is B not coercive? Show that this is true on $H^1(\Omega)$.
However, restricting how negative c may be, show that B is still coercive on $H_0^1(\Omega)$

(**c**) Consider the Neumann boundary value problem

$$\begin{cases} -\nabla \cdot (\nabla u + bu) + cu = f & \text{in } \Omega\,, \\ (-\nabla u + b\,u) \cdot \nu = g & \text{on } \partial\Omega\,, \end{cases}$$

with $\nu = \nu(x)$ the outer unit boundary normal vector at $x \in \partial\Omega$. Identify Sobolev spaces and set the corresponding variational formulation $B(u,v) = \langle F, v\rangle$ for some F (F depending on f and g) so that problem has a unique solution u (*justify your answer*), and derive estimates for the solution u showing its dependence on the coefficients b and c and the corresponding Sobolev norms associated with the data f and g.

2. Use the contraction-mapping theorem to show that the Fredholm Integral Equation

$$f(x) \;=\; \phi(x) \;+\; \lambda \int_a^b K(x,y) f(y)\, dy$$

has a unique solution $f \in C([a,b])$ provided that λ is sufficiently small, wherein $\phi \in C([a,b])$ and $K \in C([a,b] \times [a,b]))$

3. Denote by $L_0^1(\mathbb{R}^3)$ be the set of all measurable functions that vanishes outside some compact set in $\mathbb{R}^3$ and that are integrable, i.e. $\int_{\mathbb{R}^3} |f| dx < \infty$. Let $\rho \in L_0^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ be a given function and $\tilde{\rho} \in \mathcal{D}'(\mathbb{R}^3)$ be the generalized function corresponding to ρ.

(**a**) Show that the classical function

$$v(x) = \int_{\mathbb{R}^3} \frac{\rho(y) dy}{4\pi |x-y|}, \qquad x \in \mathbb{R}^3\,,$$

is well defined in $L^1(\mathbb{R}^3)$

(**b**) Show that the generalized function $V \in \mathcal{D}'(\mathbb{R}^3)$ corresponding to the classical function $v(x)$ defined above is a weak solution to the Poisson equation

$$-\Delta V = \tilde{\rho}(x), \qquad \text{in } \mathbb{R}^3.$$