Numerical Analysis Prelim Exam: Part A

Jan, 2012

1. Let $f \in C^\infty(\mathbb{R})$, bounded with compact support, and

$$g(x) := f(x) + \frac{1}{10}\sin(10\pi x).$$

   (a) Find an even polynomial $K(x)$ satisfying

$$\int_{-1}^{1} x^k K(x)dx = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, 3, \end{cases}$$

   and $K(\pm 1) = 0$.

   (b) For $\epsilon > 0$, define

$$K_\epsilon(x) := \begin{cases} \frac{1}{\epsilon}K(\frac{x}{\epsilon}), & -\epsilon \le x \le \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that there is a constant C such that for sufficiently small ϵ,

$$|f(x) - K_\epsilon * g(x)| \le C\epsilon^4.$$

2. Let $\theta \in (0, 1)$.

   (a) Determine α, β, and γ such that the quadrature $\alpha f(1) + \beta f(\theta) + \gamma f(0)$ yields the exact value of

$$\int_0^1 f(x)dx$$

   for all quadratic polynomials $f(x)$.

   (b) Define

$$f_\theta(x) = \begin{cases} 1, & \text{if } x \le \theta^2, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that with the above choice of α, β, and γ,

$$\lim_{\theta \to 0^+} \left| \int_0^1 f_\theta(x)dx - (\alpha f_\theta(1) + \beta f_\theta(\theta) + \gamma f_\theta(0)) \right| = \infty.$$

3. Let A be a real, positive definite, self-adjoint matrix. Define the energy

$$F(y) := \frac{1}{2}(x-y)^T A(x-y).$$

Consider the iterative scheme

$$x^{n+1} = x^n - s_n r_n,$$

where

$$s_n := \frac{||r_n||^2}{r_n^T A r_n}, r_n := Ax^n - b. \tag{1}$$

(a) Show that given x^n, the choice of s^n in (1) minimizes

$$E(x^{n+1}) := \frac{1}{2}(x^{n+1})^T A x^{n+1} - (x^{n+1})^T b.$$

(b) Show that $F(x^n)$ tends to 0 as n tends to ∞, and therefore x^n converges to the solution of $Ax = b$.