Numerical Analysis Prelim Exam: Part B

August, 2011

1. Consider the energy

$$E(u) := \frac{1}{2}\int_0^1 |u_x(x)|^2 dx + \frac{\lambda}{2}\int_0^1 (u(x) - f(x))^2 dx, \lambda > 0,$$

defined for $u \in C^2([0,1];\mathbb{R})$, $f \in C([0,1];\mathbb{R})$, and $u(0) = u(1) = f(0) = f(1) = 0$. Consider a discrete approximation of E as follows: $U = (u_1, u_2, \cdots, u_{N-1})^T$,

$$E_h(U) := \frac{1}{2}\sum_{j=1}^{N-1} |D^+ u_j|^2 h + \frac{\lambda}{2}\sum_{j=1}^{N-1} |u_j - f_j|^2 h,$$

where $h = 1/N$, $u_0 = u_N = 0$, $D_+ u_j = (u_{j+1} - u_j)/h$, and $f_j = f(jh)$.

   (a) Derive the linear system

$$AU = b \tag{1}$$

   whose solution minimizes E_h.

   (b) Derive the Gauss-Seidel method for this linear system and show that the iteration method will converge.

   (c) Does the solution of (1) approximate the minimizer of E? Justify your answer.

2. Consider

$$u_t = a(x)u_x, \quad 0 < x < 1, t > 0.$$

   (a) Derive an upwind scheme for the equation. Determine a suitable boundary condition such that the PDE is well-posed. Introduce a suitable discrete L^2 norm $||\cdot||_h$ and show that the upwind scheme is stable in this norm.

   (b) Derive a discontinuous Galerkin method with piecewise linear basis functions for the equation above.

3. Derive the order of accuracy of the multistep method

$$y_{n+1} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n), h > 0$$

for approximation of the solutions of $y' = y$. Is the scheme convergent? Why?