THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Jan 9, 2012

Problem 1 (30pts)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be an iid sequence of random variables. Show that the following two properties are equivalent:

(1) the distribution of X_1 is unbounded from above, i.e., $\mathbb{P}[X_1 \leq x] < 1$, for all $x \in \mathbb{R}$.
(2) $\limsup_n X_n = +\infty$, a.s.

Problem 2 (35pts)**.** Let $(S, \mathcal{S}, \nu)$ be a probability space, and let $K : S \to \mathcal{B}(\mathbb{R})$ be a kernel from $(S, \mathcal{S})$ to $\mathcal{B}(\mathbb{R})$ such that, for each $x \in S$, $\mu_x = K(x, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R})$. We define the **mixture of** $\{\mu_x\}_{x\in S}$ **by** ν to be the probability measure μ on $\mathcal{B}(\mathbb{R})$ given by

$$\mu(B) = \int_S K(x, B)\,\nu(dx), \text{ for } B \in \mathcal{B}(\mathbb{R}).$$

Derive an expression for the characteristic function of μ, using the measure ν and the characteristic functions φ_{μ_x} of the probability measures $\{\mu_x\}_{x\in S}$.
(*Hint:* Approximate the function $\xi \mapsto e^{it\xi}$ by a suitable sequence of step functions.)

Problem 3 (35pts)**.** Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a simple random walk, i.e., $X_0 = 0$, $X_n = \sum_{k=1}^n \xi_k$, for $n \in \mathbb{N}$, where $\{\xi_n\}_{n\in\mathbb{N}}$ is an iid sequence with $\mathbb{P}[\xi_1 = -1] = \mathbb{P}[\xi_1 = 1] = \frac{1}{2}$. Let the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be given by $\mathcal{F}_0 = 0$, $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$, for $n \in \mathbb{N}$.

Let $|X| = M + A$ be the Doob-Meyer decomposition of the submartingale $|X|$, into a martingale M with $M_0 = 0$ and a non-decreasing predictable process A. Show that M admits the form

$$M = H \cdot X, \tag{1}$$

for some predictable process H and find an explicit expression for H.