Preliminary Examination in Topology: January 2012

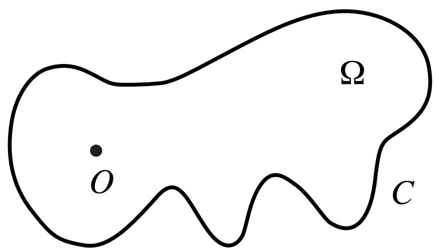
Differential Topology

Instructions: Do all three questions.

Time Limit: 90 minutes

1.

(a). Let $C \subset R^2$ be a (smooth) simple closed curve in the plane and Ω the closed subset that it bounds. Thus Ω is a 2-manifold with boundary C. Assume the origin, $O \in R^2$, lies in the interior of Ω.



Introduce polar coordinates r, θ and prove that

$$\text{Area}(\Omega) = \int_C \frac{1}{2} r^2 d\theta$$

Discuss why the integrand is well-defined and describe the orientations you use.

(b). Let $f : S^1 \to R^2 - \{0\}$ be a smooth map. Define the winding number of f about 0 and prove that it equals $\frac{1}{2\pi} \int_{S^1} f^*(d\theta)$.

(c). Suppose $\omega \in \Omega^1(S^2)$ is a 1-form with $dw = 0$. Let $f : S^1 \to S^2$ be a smooth map. Prove $\int_{S^1} f^*\omega = 0$.

2. Let $S^2 \subset E^3$ be the unit 2-sphere in Euclidean 3-space. Set $M = \{p_1, p_2 \in S^2 : p_1 \neq p_2\}$.

(a). Prove that M is a manifold. What is the dimension of M?

(b). Define $f : M \to R$ by setting $f(p_1, p_2)$ to be the **square** of the Euclidean distance between $p_1, p_2 \in S^2 \subset E^3$. Prove that f is smooth and that $S = f^{-1}(1/2) \subset M$ is a submanifold.

(c). What is the dimension of S? Is S compact? *Bonus: Can you identify S?*

3. Define $f : R^4 \to R^4$ to be $f(x, y, z, w) = (x - y, 2y, 3z, 4w)$. f induces a map $\hat{f} : RP^3 \to RP^3$.

(a). Find the fixed points of $\hat{f}$.

(b). Compute the local Lefschetz numbers of $\hat{f}$ and the global Lefschetz number of $\hat{f}$.

(c). Use the above to find the Euler characteristic of RP^3.