Preliminary Examination in Algebra–Fall semester

August 17, 2012, RLM 9.166, 1:00-2:30 p.m.

Do three of the following four problems.

1. Let G be a group, $\mathrm{Aut}(G)$ the group of all automorphisms $\varphi : G \to G$, and

$$\mathrm{Z}(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$$

the center of G.

(i.) For each h in G define a map $\psi_h : G \to G$ by

$$\psi_h(g) = hgh^{-1}.$$

Show that each map ψ_h is an element of $\mathrm{Aut}(G)$, and the set $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ of all such automorphisms is a normal subgroup of $\mathrm{Aut}(G)$.

(ii.) Prove that $\mathrm{Inn}(G)$ is isomorphic to the quotient group $G/\mathrm{Z}(G)$.

(iii.) Assume that G is a finite abelian group such that the number $|\mathrm{Aut}(G)|$ of elements in $\mathrm{Aut}(G)$ is odd. Prove that $|G|$ is either 1 or 2.

2. Let n be a square free integer greater than 3, and let R be the ring $\mathbb{Z}[x]/\langle x^2+n\rangle$, where $\langle x^2+n\rangle$ is the principal ideal generated by x^2+n.

(i.) Show that each of the elements 2, x, and $1+x$, is irreducible in R.

(ii.) Show that R is not a unique factorization domain.

(iii.) Give an example of an ideal in R that is not principal.

3. A commutative ring A is called *Artinian* if every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

is eventually constant.

(i.) Show that an Artinian integral domain is a field.

(ii.) Show that in an Artinian ring A, every prime ideal is maximal.

4. Let K be an algebraically closed field and K^N the K-vector space of $N \times 1$ column vectors with entries in K. Let A be an $N \times N$ matrix with entries in K. We say that a vector $\mathbf{v}$ in K^N is a *cyclic vector* for A if the set

$$\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, A^3\mathbf{v}, \ldots, A^{N-1}\mathbf{v}\}$$

is a basis for K^N. Prove that A has a cyclic vector if and only if the characteristic polynomial for A and the minimal polynomial for A are equal.