Preliminary Examination in Algebra–Spring semester
August 17, 2012, RLM 9.166, 2:40-4:10 p.m.

Do three of the following four problems.

1. Let K be a field of characteristic $p > 2$. Let $K(x)$ be the field of rational functions with coefficients in K, and $\mathrm{Aut}(K(x)/K)$ the group of automorphisms $\varphi : K(x) \to K(x)$ that fix the subfield K.

   (i) Show that there exist automorphisms σ and τ in $\mathrm{Aut}(K(x)/K)$ such that
$$\sigma(x) = x + 1, \qquad \text{and} \quad \tau(x) = -x.$$

   (ii) Let $H \subseteq \mathrm{Aut}(K(x)/K)$ be the subgroup generated by σ and τ. Show that the subfield $K(x)^H \subseteq K(x)$ fixed by elements of H is $K(y)$, where
$$y = (x^p - x)^2.$$

2. Suppose that $f(x)$ is a monic, irreducible polynomial in $\mathbb{Q}[x]$, but f is not cyclotomic and f is not x. Let $\alpha_1, \alpha_2, \ldots, \alpha_L$ be the distinct roots of f in $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. For each positive integer n define
$$g_n(x) = \prod_{l=1}^{L} (x - \alpha_l^n).$$

   (i) Prove that $g_n(x)$ is a positive integer power of a single irreducible polynomial in $\mathbb{Q}[x]$.

   (ii) Prove that if $1 \le m < n$ then $g_m(x)$ and $g_n(x)$ have no common zeros in $\overline{\mathbb{Q}}$.

3. Let p be a prime and $\mathbb{F}_p$ the finite field with p elements. Let $E = \mathbb{F}_p(x, y)$ and $F = \mathbb{F}_p(x^p, y^p)$, where x and y are independent indeterminants.

   (i) Determine the degree of the field extension E/F.

   (ii) Prove that E/F is not a simple extension by exhibiting infinitely many intermediate fields K such that $F \subseteq K \subseteq E$.

4. Let $K/\mathbb{Q}$ be a finite extension of fields. Assume that K is the splitting field for the polynomial $f(x) = x^4 - 3x^2 + 5$ over $\mathbb{Q}$.

   (i) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (ii) Prove that K has degree 8 over $\mathbb{Q}$.

   (iii) Determine the Galois group of the extension $K/\mathbb{Q}$ and show how it acts on the roots of f.